1. Let A, B, C, and u be

$$A = \begin{pmatrix} 3 & 7 & -2 & 1 \\ 1 & 4 & 5 & 5 \\ 0 & 2 & 6 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 & 1 \\ 3 & 7 & 6 \end{pmatrix}, \quad C = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}, \quad u = \begin{pmatrix} 4 \\ -2 \\ 1 \\ -1 \end{pmatrix}$$

Compute the following :

a.(3pts) Au

$$Au = \begin{pmatrix} 12-14-2-1 \\ 4-8+5-5 \\ -4+6-1 \end{pmatrix} = \begin{pmatrix} -5 \\ -4 \\ 1 \end{pmatrix}$$

b.(3pts) A^T

$$A^T = \begin{pmatrix} 3 & 1 & 0 \\ 7 & 4 & 2 \\ -2 & 5 & 6 \\ 1 & 5 & 1 \end{pmatrix}$$

c.(4pts) BC^T

$$BC^T = \begin{pmatrix} 1 & 1 & 1 \\ 3 & 7 & 6 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} 2-1 & 1-1 \\ 6-6 & 7-6 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

2. For the given matrix M

$$\begin{pmatrix} 1 & 3 & -2 & 4 & 5 \\ 3 & 0 & -6 & 7 & -2 \\ 2 & 1 & 0 & 3 & 5 \\ 2 & 3 & 8 & 5 & 7 \end{pmatrix}$$

answer the two questions below.

a. (7pts) Find the reduced echelon form of M and a basis of Col M. Find m such that Col $M \subset \mathbb{R}^m$ and compute dim Col M.

$$\begin{pmatrix} 1 & 3 & -2 & 4 & 5 \\ 3 & 0 & -6 & 7 & -2 \\ 2 & 1 & 0 & 3 & 5 \\ 2 & 3 & 8 & 5 & 7 \end{pmatrix} \underset{(4)\to(4)-(3)}{\sim} \begin{pmatrix} 1 & 3 & -2 & 4 & 5 \\ 3 & 0 & -6 & 7 & -2 \\ 2 & 1 & 0 & 3 & 5 \\ 0 & 2 & 8 & 2 & 2 \end{pmatrix} \underset{(4)\to(4)/2}{\sim} \begin{pmatrix} 1 & 3 & -2 & 4 & 5 \\ 3 & 0 & -6 & 7 & -2 \\ 2 & 1 & 0 & 3 & 5 \\ 0 & 1 & 4 & 1 & 1 \end{pmatrix}$$

$$\underset{(2)\to(2)-(1)\times3}{\sim} \begin{pmatrix} 1 & 3 & -2 & 4 & 5 \\ 0 & -9 & 0 & -5 & -17 \\ 2 & 1 & 0 & 3 & 5 \\ 0 & 1 & 4 & 1 & 1 \end{pmatrix} \underset{(3)\to(3)-(1)\times2}{\sim} \begin{pmatrix} 1 & 3 & -2 & 4 & 5 \\ 0 & -9 & 0 & -5 & -17 \\ 0 & -5 & 4 & -5 & -5 \\ 0 & 1 & 4 & 1 & 1 \end{pmatrix} \underset{(3)\to(3)+(4)}{\sim} \begin{pmatrix} 1 & 3 & -2 & 4 & 5 \\ 0 & -9 & 0 & -5 & -17 \\ 0 & -4 & 8 & -4 & -4 \\ 0 & 1 & 4 & 1 & 1 \end{pmatrix}$$

$$\underset{(3)\to(3)/4}{\sim} \begin{pmatrix} 1 & 3 & -2 & 4 & 5 \\ 0 & -9 & 0 & -5 & -17 \\ 0 & -1 & 2 & -1 & -1 \\ 0 & 1 & 4 & 1 & 1 \end{pmatrix} \underset{(3)\to(3)+(4)}{\sim} \begin{pmatrix} 1 & 3 & -2 & 4 & 5 \\ 0 & -9 & 0 & -5 & -17 \\ 0 & 0 & 6 & 0 & 0 \\ 0 & 1 & 4 & 1 & 1 \end{pmatrix} \underset{\substack{(3)\to(3)/6 \\ \text{and } (2)\to(2)+(4)}}{\sim} \begin{pmatrix} 1 & 3 & -2 & 4 & 5 \\ 0 & -8 & 4 & -4 & -16 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 4 & 1 & 1 \end{pmatrix}$$

$$\underset{(2)\to(2)/4}{\sim} \begin{pmatrix} 1 & 3 & -2 & 4 & 5 \\ 0 & -2 & 1 & -1 & -4 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 4 & 1 & 1 \end{pmatrix} \underset{(2)\to(2)+(4)\times2}{\sim} \begin{pmatrix} 1 & 3 & -2 & 4 & 5 \\ 0 & 0 & 9 & 1 & -2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 4 & 1 & 1 \end{pmatrix} \underset{(2)\leftrightarrow(4)}{\sim} \begin{pmatrix} 1 & 3 & -2 & 4 & 5 \\ 0 & 1 & 4 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 9 & 1 & -2 \end{pmatrix}$$

$$\underset{(2)\to(2)-(3)\times4}{\sim} \begin{pmatrix} 1 & 3 & -2 & 4 & 5 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 9 & 1 & -2 \end{pmatrix} \underset{(4)\to(4)-(3)\times9}{\sim} \begin{pmatrix} 1 & 3 & -2 & 4 & 5 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 \end{pmatrix} \underset{(2)\to(2)-(4)}{\sim} \begin{pmatrix} 1 & 3 & -2 & 4 & 5 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 \end{pmatrix}$$

$$\underset{\substack{(1)\to(1)-(2)\times3 \\ +(3)\times2-(4)\times4}}{\sim} \begin{pmatrix} 1 & 0 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 \end{pmatrix}$$: The reduced echelon form of M.

Since the first four columns are pivot columns, $\left\{ \begin{pmatrix} 1 \\ 3 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} -2 \\ -6 \\ 0 \\ 8 \end{pmatrix}, \begin{pmatrix} 4 \\ 7 \\ 3 \\ 5 \end{pmatrix} \right\}$ is a basis for Col M.

$m = 4$ since the matrix is 4×5. dim Col M = rank M = # (pivot columns) = 4.

b. (5pts) Define M_l as the matrix obtained from M by deleting the 5th column of M. Find M_l^{-1} and $\det M_l$.

Lets first write down what elementary operations we have done.

$(4) \xrightarrow{R} (4)-(3)$, $(4) \xrightarrow{S} (4)/2$, $(2) \xrightarrow{R} (2)-(1)\times 3$, $(3) \xrightarrow{R} (3)-(1)\times 2$,

$(3) \xrightarrow{R} (3)+(4)$, $(3) \xrightarrow{S} (3)/4$, $(3) \xrightarrow{R} (3)+(4)$, $(3) \xrightarrow{S} (3)/6$,

$(2) \xrightarrow{R} (2)+(4)$, $(2) \xrightarrow{S} (2)/4$, $(2) \xrightarrow{R} (2)+(4)\times 2$, $(2) \xleftrightarrow{I} (4)$,

$(2) \xrightarrow{R} (2)-(3)\times 4$, $(4) \xrightarrow{R} (4)-(3)$, $(2) \xrightarrow{R} (2)-(4)$, $(1) \xrightarrow{R} (1)-(2)\times 3+(3)\times 2-(4)\times 4$.

R (replacement) does not change the determinant.
I (interchange) changes the determinant into itself multiplied by -1.
S (scaling) changes the determinant into itself multiplied by k: scaling factor.
Hence, the determinant of the identity matrix is
the determinant of M_l multiplied by $\frac{1}{2}\cdot\frac{1}{4}\cdot\frac{1}{6}\cdot\frac{1}{4}\cdot(-1)$

$$= -\frac{1}{192}.$$

$$\therefore \det M_l = \underbrace{-192}$$

In order to get the inverse matrix, lets apply elementary ~~oper~~ row operations written in the above,

$$\begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix} \to \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&-1&1 \end{pmatrix} \to \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&-\frac12&\frac12 \end{pmatrix} \to \begin{pmatrix} & '' & & \\ -3&1&0&0\\ & '' & & \\ & '' & & \end{pmatrix} \to \begin{pmatrix} & '' & & \\ & '' & & \\ -2&0&1&0\\ & '' & & \end{pmatrix} \to \begin{pmatrix} & '' & & \\ & '' & & \\ -2&0&\frac12&\frac12\\ & '' & & \end{pmatrix}$$

$$\to \begin{pmatrix} & '' & & \\ & '' & & \\ -\frac12&0&\frac18&\frac18\\ & '' & & \end{pmatrix} \to \begin{pmatrix} & '' & & \\ & '' & & \\ -\frac12&0&-\frac38&\frac58\\ & '' & & \end{pmatrix} \to \begin{pmatrix} & '' & & \\ & '' & & \\ -\frac1{12}&0&-\frac1{16}&\frac5{48}\\ & '' & & \end{pmatrix} \to \begin{pmatrix} & '' & & \\ -3&1&-\frac12&\frac12\\ & '' & & \\ & '' & & \end{pmatrix} \to \begin{pmatrix} & '' & & \\ -\frac34&\frac14&-\frac18&\frac18\\ & '' & & \\ & '' & & \end{pmatrix}$$

$$\to \begin{pmatrix} & '' & & \\ -\frac34&\frac14&-\frac98&\frac98\\ & '' & & \\ & '' & & \end{pmatrix} \to \begin{pmatrix} & '' & & \\ 0&0&-\frac12&\frac12\\ & '' & & \\ -\frac34&\frac14&-\frac98&\frac98 \end{pmatrix} \to \begin{pmatrix} & '' & & \\ \frac13&0&-\frac14&\frac1{12}\\ & '' & & \\ & '' & & \end{pmatrix} \to \begin{pmatrix} & '' & & \\ & '' & & \\ & '' & & \\ 0&\frac14&-\frac9{16}&\frac3{16} \end{pmatrix} \to \begin{pmatrix} & '' & & \\ \frac13&-\frac14&\frac5{16}&-\frac5{48}\\ & '' & & \\ & '' & & \end{pmatrix}$$

$$\to \begin{pmatrix} -\frac16&-\frac14&\frac{19}{16}&-\frac{11}{48}\\ \frac13&-\frac14&\frac5{16}&-\frac5{48}\\ -\frac1{12}&0&-\frac1{16}&\frac5{48}\\ 0&\frac14&-\frac9{16}&\frac3{16} \end{pmatrix} = \frac{1}{48}\begin{pmatrix} -8&-12&57&-11\\ 16&-12&15&-5\\ -4&0&-3&5\\ 0&12&-27&27 \end{pmatrix} = M_l^{-1}$$

3. Mark each statement True or False. Justify your answer precisely. (You can use any theorems or definitions you have learned in class or in the book. *Extra credit to those who answer all the true/false questions and justify them correctly.*)

a. (3pts) For a matrix A, there exists a unique echelon form of A.

False : there exists a unique REDUCED echelon form of A.
(Ex. $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ NOT UNIQUE)

b. (3pts) The rank of an $m \times n$ matrix A is exactly same as the number of pivot columns of the reduced echelon form of A.

True : Since row operations do not change the linear dependence relations of columns, the maximal number of vectors which are linearly independent should be the same under row operations. It means that the number of vectors in a basis of Col A and Col (reduced echelon form of A) are the same.

c. (3pts) Suppose that six vectors $v_1, v_2, \cdots, v_6$ satisfy :

$$\{v_1, v_2, v_3, v_4\},\ \{v_3, v_4, v_5, v_6\},\text{ and } \{v_5, v_6, v_1, v_2\} \text{ are linearly independent sets of vectors.}$$

Then, $\{v_1, v_2, v_3, v_4, v_5, v_6\}$ is a linearly independent set.

False : Counterexample.

$$\underset{v_1}{\underset{\parallel}{e_1}},\ \underset{v_2}{\underset{\parallel}{e_2}},\ \underset{v_3}{\underset{\parallel}{e_3}},\ \underset{v_4}{\underset{\parallel}{e_4}},\ \underset{v_5}{\underset{\parallel}{e_1+e_3}},\ \underset{v_6}{\underset{\parallel}{e_2+e_4}} \text{ in } \mathbb{R}^4.$$

d. (3pts) For every $n \times n$ matrix A, Col A is a subspace of $\mathbb{R}^n$.

True : 1) $\vec{0} = A \cdot \vec{0} \in \text{Col } A$

2) $Au + Av = A(u+v) \in \text{Col } A$

3) $\cancel{A(cu)} \cancel{= c}$
$c \cdot Au = A(cu) \in \text{Col } A$

Hence, Col A is a subspace of $\mathbb{R}^n$.

e. (3pts) There exist two 3×3 matrices A and B such that

$$\text{Col } A \cup \text{Col } B$$

is NOT ~~a~~ subspace of $\mathbb{R}^3$.

True : $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$\Rightarrow \text{Col } A = \text{Span}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right\}$, $\text{Col } B = \text{Span}\left\{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right\}$.

In Col $A \cup$ Col B, there are two vectors $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \in \text{Col } A$
$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \in \text{Col } B$

such that $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ which cannot be included in Col $A \cup$ Col B.

f. (3pts) For every 3×3 matrix A,

$$\text{Col } A \neq \text{Nul } A.$$

True : By the rank Theorem,

$$\dim \text{Col } A + \dim \text{Nul } A = 3.$$

Therefore, $\dim \text{Col } A \neq \dim \text{Nul } A$. (they should have different parity.)

Obviously, Col $A \neq$ Nul A.

Remark (Please note that when A is a 2×2 matrix, Col A can be the same as Nul A. For instance, $A = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$.)

4. Let $\mathbb{P}_4$ be the set of all polynomials of degree at most 4. Define

$$S = \{p(x) \in \mathbb{P}_4 : p(1) = 0\}$$

a. (7pts) Show that S is a subspace of $\mathbb{P}_4$.

For S to be a subspace of $\mathbb{P}_4$, we only need to show three things.

① $0 \in S$

the zero vector in $\mathbb{P}_4$ is $p_0 = 0\cdot 1 + 0\cdot t + 0\cdot t^2 + 0\cdot t^3 + 0 t^4 = 0$. and $p_0(1) = 0$.

$\therefore$ the zero vector in in S.

② $p_1, p_2 \in S$

if $p_1(1) = 0$, $p_2(1) = 0$, then $\underbrace{p_1(1) + p_2(1)}_{(p_1+p_2)(1)} = 0$ hence $(p_1+p_2)(t) \in S$.

③ $p \in S$, $c \in \mathbb{R}$

if $p(1) = 0$, then $c\cdot p(1) = c\cdot 0 = 0$ hence $(c\cdot p)(t) \in S$.

Therefore S is a subspace of $\mathbb{P}_4$.

b. (7pts) Find a basis of S. What will $\dim S$ be?

Let's find an explicit expression for $p(t) \in S$.

In general, we can put $p(t) = a_4 t^4 + a_3 t^3 + \cdots + a_0$, for $p(t)$ to satisfy $p(1) = 0$,

$a_4 + a_3 + a_2 + a_1 + a_0 = 0$.

Hence, $a_0 = -a_1 - a_2 - a_3 - a_4$. Here, a_1, a_2, a_3, a_4 are free.

This implies that $p(t) \in S$ can be written as

$a_4(t^4-1) + a_3(t^3-1) + a_2(t^2-1) + a_1(t-1)$, $a_1, a_2, a_3, a_4 \in \mathbb{R}$.

It shows that $S = \{a_4(t^4-1) + a_3(t^3-1) + a_2(t^2-1) + a_1(t-1) \mid a_1, a_2, a_3, a_4 \in \mathbb{R}\}$.

$= \text{Span}\{\underbrace{t^4-1, t^3-1, t^2-1, t-1}_{\text{they are linearly independent for sure.}}\}$

$\therefore \{t^4-1, t^3-1, t^2-1, t-1\}$ is a basis of S.

And $\dim S = 4$.

5. **(6pts)** Let A be the following 2×2 matrix:

$$A = \begin{pmatrix} 1 & 3 \\ 0 & 5 \end{pmatrix}$$

Find all possible real numbers x such that

$$\det(A - xI) = 0$$

Here, I is the 2×2 identity matrix.

$$A - xI = \begin{pmatrix} 1 & 3 \\ 0 & 5 \end{pmatrix} - x \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1-x & 3 \\ 0 & 5-x \end{pmatrix}$$

$$\therefore \det(A - xI) = \det \begin{pmatrix} 1-x & 3 \\ 0 & 5-x \end{pmatrix} = (1-x)(5-x) - 0 \cdot 3$$
$$= (1-x)(5-x)$$

For this to be the zero, $x=1$ or $x=5$.

Answer: 1 , 5.