1. (2pts) Solve the initial value problem

$$y''(t) - 6y'(t) + 13y(t) = 0, \quad y(1) = 2,\ y'(1) = 0$$

2. (3pts) Given a linear second-order equation

$$y''(t) + ay'(t) + by(t) = f(t),$$

only information you have is a set of three solutions to the equation. They are

$$t + e^t \cos 2t + e^{2t} \sin t,\ t + e^{2t} \sin t,\ t + e^t \sin 2t + e^{2t} \sin t.$$

Find a, b, and $f(t)$.

3. (4pts) A 1-kg mass is attached to a spring with stiffness 17N/m. The damping constant for the system is 8N-sec/m. At some point, the mass was located 3m distance to the left of equilibrium and had velocity 17m/sec to right direction. What is the maximum displacement to the right that it will attain?

4. (3pts) Find a general solution to

$$y''''(t) - 4y'''(t) + 7y''(t) - 6y'(t) + 2y(t) = 0$$

5. (The Energy Integral Lemma)

a. (2pts) Let $y(t)$ be a solution to the differential equation

$$y''(t) = f(y(t))$$

where $f(y)$ is a function of y. Let $F(y)$ be an indefinite integral of $f(y)$, that is,

$$f(y) = \frac{d}{dy}F(y).$$

Show that the quantity

$$E(t) := \frac{1}{2}y'(t)^2 - F(y(t))$$

is constant. (Proving this, please give appropriate explanations for each steps and each equalities(=).)

b. (4pts) Find a solution to the given differential equation with initial values

$$y''(t) = y(t) - \frac{1}{y(t)^3}, \quad y(0) = \sqrt{2},\ y'(0) = \frac{1}{\sqrt{2}}$$

(Note that a solution does not need to be unique since the differential equation is a **non**linear second-order equation.)

6. (Relation between the Wronskian and Linear Dependence)

a. (2pts) Decide whether the statement below is True or False. If it is True, write down TRUE. If it is False, then give a counterexample.

For every $n \geq 2$, if the Wronskian

$$W[\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n](t)$$

is zero then $\mathbf{x}_1(t)$, $\mathbf{x}_2(t)$, $\cdots$, $\mathbf{x}_n(t)$ are linearly dependent.

b. (2pts) Show that three functions

$$\frac{1}{-t}, \frac{1}{1-t}, \frac{1}{2-t}$$

are linearly independent.

7. (4pts, The Method of Variation of Parameters)

Let $y(t)$ be a particular solution to

$$y'''(t) + ay''(t) + by'(t) + cy(t) = f(t)$$

where a, b, $c \in \mathbb{R}$ and $y_1(t)$, $y_2(t)$, and $y_3(t)$ be 3 linearly independent solutions to the homogeneous case. We want to find $v_1(t)$, $v_2(t)$, and $v_3(t)$ such that

$$y(t) = v_1(t)y_1(t) + v_2(t)y_2(t) + v_3(t)y_3(t).$$

By imposing two appropriate conditions, prove that

$$\begin{pmatrix} v_1'(t) \\ v_2'(t) \\ v_3'(t) \end{pmatrix} = \begin{pmatrix} y_1(t) & y_2(t) & y_3(t) \\ y_1'(t) & y_2'(t) & y_3'(t) \\ y_1''(t) & y_2''(t) & y_3''(t) \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 0 \\ f(t) \end{pmatrix}.$$

(Hint. One of the conditions you would impose is $v_1'(t)y_1(t) + v_2'(t)y_2(t) + v_3'(t)y_3(t) = 0$.)

8. (2pts) Change the differential equation

$$y'''(t) + \frac{1}{1+t}y''(t) - y(t) = \sin t$$

into the form of

$$\mathbf{x}'(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{f}(t)$$

where $\mathbf{x}(t)$, $\mathbf{f}(t)$ are 3×1 matrices and $\mathbf{A}(t)$ is a 3×3 matrix.

9. (3pts) On the interval $(-1, 1)$, let's define $y_1(t) = t^2 - 2$, $y_2(t) = \sin t$, and $y_3(t) = \cos t$. Find the Wronskian of $y_1(t)$, $y_2(t)$, and $y_3(t)$ and show that $y_1(t)$, $y_2(t)$, $y_3(t)$ are linearly independent. Can you assure the existence or non-existence of a homogeneous linear system for which $y_1(t)$, $y_2(t)$, and $y_3(t)$ are solutions? Why?

1 A mass-spring system

Given a mass-spring system

$$my'' + by' + ky = F_{ext}$$

m is the mass, b is the damping coefficient, and k is the stiffness.

Since we derived the equation from the real world, we implicitly assume that $m > 0$, $b \geq 0$, $k > 0$. When there is no friction force or $b = 0$, the system is called **undamped**. For $0 < b < \sqrt{4mk}$, the system is called **underdamped**. Meanwhile, the system is called overdamped when $b > \sqrt{4mk}$. As the last thing, when b is exactly $\sqrt{4mk}$, the system is said to be **critcally damped**. For the external force term, the system is called **free** if $F_{ext} = 0$.

For the sake of convenience, we define

$$\omega = \sqrt{\frac{k}{m}}$$

so that the equation becomes

$$y'' + \frac{b}{\sqrt{4mk}} 2\omega y' + \omega^2 y = \frac{F_{ext}}{m}$$

Undamped and Free The general solution is expressed in the form

$$A\cos(\omega t + \phi)$$

The period is $\frac{2\pi}{a}$ and the natural frequency is $\frac{a}{2\pi}$.

Underdamped and Free In this case, the general solution is expressed in the form

$$Ae^{\alpha t}\sin(\beta t + \phi)$$

The term $Ae^{\alpha t}$ is called an exponential **damping factor** and **quasiperiod** is $\frac{2\pi}{\beta}$ and **quasifrequency** is $\frac{\beta}{2\pi}$.

Overdamped and Free The general solution has the form as

$$c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

Critically damped and Free The general solution has the form as

$$(c_1 + c_2 t)e^{rt}$$

2 Vandermonde determinant

The determinant of

$$\begin{pmatrix} 1 & \alpha & \alpha^2 \\ 1 & \beta & \beta^2 \\ 1 & \gamma & \gamma^2 \end{pmatrix}$$

is $(\alpha - \beta)(\alpha - \gamma)(\beta - \gamma)$.

3 The integral of a specific function

For a function

$$\frac{y}{y^2 + 1},$$

it would be a good way to use the substitution method with $s = y^2 + 1$ in order to integrate the function

.