1. a. (5pts) In $\mathbb{R}^5$, you are given 3 vectors

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \; v_2 = \begin{pmatrix} 3 \\ 1 \\ -1 \\ 0 \\ 2 \end{pmatrix}, \; v_3 = \begin{pmatrix} 7 \\ -2 \\ 1 \\ 1 \\ 3 \end{pmatrix}$$

Apply Gram-Schmidt (Orthogonalization) Process to find an orthonormal basis of Span$\{v_1, v_2, v_3\}$.

$$u_1 = v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad u_2 = v_2 - \frac{\langle u_1, v_2 \rangle}{\langle u_1, u_1 \rangle} u_1 = \begin{pmatrix} 3 \\ 1 \\ -1 \\ 0 \\ 2 \end{pmatrix} - \frac{5}{5} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ -2 \\ -1 \\ 1 \end{pmatrix}$$

$$u_3 = v_3 - \frac{\langle u_1, v_3 \rangle}{\langle u_1, u_1 \rangle} u_1 - \frac{\langle u_2, v_3 \rangle}{\langle u_2, u_2 \rangle} u_2 = \begin{pmatrix} 7 \\ -2 \\ 1 \\ 1 \\ 3 \end{pmatrix} - \frac{10}{5} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} - \frac{14}{10} \begin{pmatrix} 2 \\ 0 \\ -2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ -4 \\ -1 \\ -1 \\ 1 \end{pmatrix} - \frac{1}{5} \begin{pmatrix} 14 \\ 0 \\ 14 \\ -7 \\ 7 \end{pmatrix}$$

$$= \frac{1}{5} \begin{pmatrix} 11 \\ -20 \\ 9 \\ 2 \\ -2 \end{pmatrix}$$

$\{u_1, u_2, u_3\}$ is an orthogonal basis.

To change it to an orthonormal basis, we divide them by their lengths.

$$\Rightarrow \frac{1}{\sqrt{5}} u_1, \; \frac{1}{\sqrt{10}} u_2, \; \frac{1}{\sqrt{11^2+20^2+9^2+2^2+2^2}} \begin{pmatrix} 11 \\ -20 \\ 9 \\ 2 \\ -2 \end{pmatrix}$$

$\sqrt{610}$

b. (8pts) Solve the least-squares problem

$$A\mathbf{x} = \mathbf{b} \text{ where } A = \begin{pmatrix} 1 & 3 & 7 \\ 1 & 1 & -2 \\ 1 & -1 & 1 \\ 1 & 0 & 1 \\ 1 & 2 & 3 \end{pmatrix}, \; \mathbf{b} = \begin{pmatrix} 2 \\ 1 \\ 2 \\ -5 \\ 5 \end{pmatrix}$$

in two different ways. (Hint. One way is to use the result of **a**. For any theorems you might use, please state them correctly, though you do not need to prove the theorems.)

① A least-square solution x is the vector which makes Ax be the orthogonal projection of b.

~~If $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, then~~ b's orthogonal projection using an orthogonal basis $\left\{ \underbrace{\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}}_{u_1}, \underbrace{\begin{pmatrix} 2 \\ 0 \\ -2 \\ -1 \\ 1 \end{pmatrix}}_{u_2}, \underbrace{\begin{pmatrix} 11 \\ -20 \\ 9 \\ 2 \\ -2 \end{pmatrix}}_{u_3'} \right\}$

is $$\frac{5}{5} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + \frac{10}{10} \begin{pmatrix} 2 \\ 0 \\ -2 \\ -1 \\ 1 \end{pmatrix} + \frac{0}{610} \begin{pmatrix} 11 \\ -20 \\ 9 \\ 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ -1 \\ 0 \\ 2 \end{pmatrix}$$

Hence, $x = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

② We can find the answer using the normal equation.

$$A^TAx = A^Tb$$

$$A^TA = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 3 & 1 & -1 & 0 & 2 \\ 7 & -2 & 1 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 3 & 7 \\ 1 & 1 & -2 \\ 1 & -1 & 1 \\ 1 & 0 & 1 \\ 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 5 & 5 & 10 \\ 5 & 15 & 24 \\ 10 & 24 & 64 \end{pmatrix} \qquad A^Tb = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 3 & 1 & -1 & 0 & 2 \\ 7 & -2 & 1 & 1 & 3 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 2 \\ -5 \\ 5 \end{pmatrix} = \begin{pmatrix} 5 \\ 15 \\ 24 \end{pmatrix}$$

$$\Rightarrow x = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

2. Define $\mathcal{B} = \left\{ \begin{pmatrix} 5 \\ 5 \\ -3 \end{pmatrix}, \begin{pmatrix} -3 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \right\}$.

a. (3pts) Find the inverse matrix of P where

$$P = \begin{pmatrix} 5 & -3 & 1 \\ 5 & 2 & 2 \\ -3 & 0 & -1 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 5 & -3 & 1 & 1 & 0 & 0 \\ 5 & 2 & 2 & 0 & 1 & 0 \\ -3 & 0 & -1 & 0 & 0 & 1 \end{array}\right) \sim \begin{pmatrix} 5 & -3 & 1 & 1 & 0 & 0 \\ 0 & 5 & 1 & -1 & +1 & 0 \\ -3 & 0 & -1 & 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} -3 & 0 & -1 & 0 & 0 & 1 \\ 0 & 5 & 1 & -1 & +1 & 0 \\ 5 & -3 & 1 & 1 & 0 & 0 \end{pmatrix}$$

$$\sim \begin{pmatrix} -1 & 0 & -\frac{1}{3} & 0 & 0 & \frac{1}{3} \\ 0 & 1 & \frac{1}{5} & -\frac{1}{5} & \frac{1}{5} & 0 \\ 5 & -3 & 1 & 1 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & \frac{1}{3} & 0 & 0 & -\frac{1}{3} \\ 0 & 1 & \frac{1}{5} & -\frac{1}{5} & \frac{1}{5} & 0 \\ 0 & 0 & \frac{3}{5}+1(-\frac{5}{3}) & & & \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & \frac{1}{3} & 0 & 0 & -\frac{1}{3} \\ 0 & 1 & \frac{1}{5} & -\frac{1}{5} & \frac{1}{5} & 0 \\ 0 & 0 & -\frac{1}{15} & \frac{2}{5} & \frac{3}{5} & \frac{5}{3} \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & 0 & 2 & 3 & 8 \\ 0 & 1 & 0 & 1 & 2 & 5 \\ 0 & 0 & 1 & -6 & -9 & -25 \end{pmatrix} \Rightarrow P^{-1} = \begin{pmatrix} 2 & 3 & 8 \\ 1 & 2 & 5 \\ -6 & -9 & -25 \end{pmatrix}$$

b. (4pts) Find the $\mathcal{B}$-coordinate of $\begin{pmatrix} 4 \\ 2 \\ -11 \end{pmatrix}$.

$$= \begin{pmatrix} 2 & 3 & 8 \\ 1 & 2 & 5 \\ -6 & -9 & -25 \end{pmatrix} \begin{pmatrix} 4 \\ 2 \\ -11 \end{pmatrix} = \begin{pmatrix} -74 \\ -47 \\ 233 \end{pmatrix}$$

c. (5pts) Let a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the map sending $\mathbf{x}$ to $A\mathbf{x}$ where

$$A = \begin{pmatrix} 12 & 15 & 40 \\ 10 & 17 & 40 \\ -6 & -9 & -22 \end{pmatrix}$$

Find the $\mathcal{B}$-matrix for T.

$$P^{-1}AP = \begin{pmatrix} 2 & 3 & 8 \\ 1 & 2 & 5 \\ -6 & -9 & -25 \end{pmatrix} \begin{pmatrix} 12 & 15 & 40 \\ 10 & 17 & 40 \\ -6 & -9 & -22 \end{pmatrix} \begin{pmatrix} 5 & -3 & 1 \\ 5 & 2 & 2 \\ -3 & 0 & -1 \end{pmatrix}$$

$$= P^{-1} \begin{pmatrix} 15 & -6 & 2 \\ 15 & 4 & 4 \\ -9 & 0 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

As long as you find that

$A\begin{pmatrix} 5 \\ 5 \\ -3 \end{pmatrix} = \begin{pmatrix} 15 \\ 15 \\ -9 \end{pmatrix} = 3\begin{pmatrix} 5 \\ 5 \\ -3 \end{pmatrix}$ (this equality)

and $\begin{pmatrix} -6 \\ 4 \\ 0 \end{pmatrix} = 2\begin{pmatrix} -3 \\ 2 \\ 0 \end{pmatrix}$, ... ,

you do not need calculations for this multiplication. Since, $P^{-1}AP = \underline{D}$ for D s.t. its diagonal entries are eigenvalues.

3. Write "TRUE" if the statement is always true, "FALSE" if it is sometimes false. ***No explanations are needed.***

a. (2pts) Given a subspace W of V, the orthogonal projection map from V to W is a one-to-one linear transformation.

False. (It does not need to be one-to-one.)

b. (2pts) The orthogonal complement of the null space of A is the same as the column space of A if A is symmetric.

True. $\left((\mathrm{Nul}\,A)^{\perp} = \mathrm{Row}\,A = \mathrm{Col}(A^T) = \mathrm{Col}(A) \right)$

c. (2pts) If the orthogonal complement of the null space of A is the same as the column space of A, then A is symmetric.

False. (Every invertible matrix can be a counterexample.)

d. (2pts) The quadratic form Q on $\mathbb{R}^3$ defined as

$$Q(x_1, x_2, x_3) = 3x_1^2 + 2x_2^2 + x_3^2 + 4x_1x_2 - 4x_2x_3$$

is an indefinite quadratic form.

$\cancel{\begin{pmatrix} 3 & 2 & 0 \\ 2 & 2 & -2 \\ 0 & -2 & 1 \end{pmatrix}}$ $\begin{pmatrix} Q(1,0,0) = 3 > 0 \\ Q(0,1,1) = -1 < 0 \end{pmatrix}$ True.

e. (2pts) Let a vector space $\mathbb{R}^3$ be equipped with an inner product defined as

$$\langle (x_1, x_2, x_3), (y_1, y_2, y_3) \rangle = (x_1 + 2x_2)(y_1 + 2y_2) + x_3y_3$$

In this inner product space, $(1, 0, 0)$ and $(0, 1, 0)$ are orthogonal still.

False. $\left(\langle (1,0,0), (0,1,0) \rangle = (1)(2) + 0 = 2 \neq 0 \right)$

f. (2pts) A square matrix A is invertible if and only if 0 is not an eigenvalue of A.

True. (A : invertible $\Leftrightarrow \det A \neq 0$
$\Leftrightarrow \det(A - 0 \cdot I) \neq 0$.)

4. (8pts) Find the maximum and minimum values of

$$Q(x_1, x_2, x_3) = -x_1^2 + x_2^2 - 7x_3^2 - 8x_1x_2 - 8x_2x_3$$

subject to the constraint

$$x_1^2 + x_2^2 + x_3^2 = 1$$

The matrix of the quadratic form Q, is

$$A = \begin{pmatrix} -1 & -4 & 0 \\ -4 & 1 & -4 \\ 0 & -4 & -7 \end{pmatrix}.$$

$$\chi_A(\lambda) = \det(A - \lambda I)$$
$$= \det\begin{pmatrix} -1-\lambda & -4 & 0 \\ -4 & 1-\lambda & -4 \\ 0 & -4 & -7-\lambda \end{pmatrix} = (-1-\lambda)\Big((1-\lambda)(-7-\lambda) - 16\Big) - (-4)\cdot(-4)(-7-\lambda).$$
$$= -(\lambda+1)\left(\lambda^2+6\lambda-7-16\right) + 16\lambda + 112$$
$$= -(\lambda^3+7\lambda^2-17\lambda-23) + 16\lambda + 112$$
$$= -\lambda^3 - 7\lambda^2 + 33\lambda + 135$$

plug in 5 first: $5\cdot(-25-35+33+27) = 5\cdot 0.$

$$= -(\lambda-5)(\lambda^2+12\lambda+27)$$
$$= -(\lambda-5)(\lambda+3)(\lambda+9).$$

So, the maximum is 5
minimum is -9.

5. Let A be

$$\begin{pmatrix} 3 & -4 & -4 \\ 2 & 1 & -4 \\ -2 & 0 & 5 \end{pmatrix}$$

whose characteristic polynomial $\chi_A(\lambda)$ is $-(\lambda-1)(\lambda-3)(\lambda-5)$.

a. (5pts) Find 3 linearly independent eigenvectors and, using them, find a diagonal matrix D and an invertible matrix P such that

$$P^{-1}AP = D$$

We have 3 distinct eigenvalues $\Rightarrow$ we can find 3 linearly independent eigenvectors.

$$\mathrm{Nul}\,(A-I) = \mathrm{Nul}\begin{pmatrix} 2 & -4 & -4 \\ 2 & 0 & -4 \\ -2 & 0 & 4 \end{pmatrix} = \mathrm{Span}\left\{\begin{pmatrix} 2 \\ 0 \\ +1 \end{pmatrix}\right\}$$

$$\mathrm{Nul}\,(A-3I) = \mathrm{Nul}\begin{pmatrix} 0 & -4 & -4 \\ 2 & -2 & -4 \\ -2 & 0 & 2 \end{pmatrix} = \mathrm{Span}\left\{\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}\right\}$$

$$\mathrm{Nul}\,(A-5I) = \mathrm{Nul}\begin{pmatrix} -2 & -4 & -4 \\ 2 & -4 & -4 \\ -2 & 0 & 0 \end{pmatrix} = \mathrm{Span}\left\{\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}\right\}$$

$\therefore$ If we let P be $\begin{pmatrix} 2 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$, then $P^{-1}AP = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix} = D$

b. (6pts) You have found only one pair of (D, P) in problem **a**. Find all possible D's. For each D, find one corresponding invertible matrix P such that $P^{-1}AP = D$.

For the equation $P^{-1}AP=D$ ~~to be~~ to be true, columns of P ~~are~~ should be eigenvectors of A or the zero vector. However, since P is invertible they should be eigenvectors. As a result, D's diagonal entries should be corresponding eigenvalues. Hence there could be 6 D's.

① $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}$, $P = \begin{pmatrix} 2 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$ ② $D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{pmatrix}$, $P = \begin{pmatrix} 1 & 2 & 0 \\ -1 & 0 & 1 \\ 1 & 1 & -1 \end{pmatrix}$

③ $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 3 \end{pmatrix}$, $P = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$ ④ $D = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$, $P = \begin{pmatrix} 0 & 2 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 1 \end{pmatrix}$

⑤ $D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $P = \begin{pmatrix} 1 & 0 & 2 \\ -1 & +1 & 0 \\ +1 & -1 & 1 \end{pmatrix}$ ⑥ $D = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $P = \begin{pmatrix} 0 & 1 & 2 \\ 1 & -1 & 0 \\ -1 & 1 & 1 \end{pmatrix}$.

6. Let T be a transformation from $\mathbb{P}_2$ to $\mathbb{R}^3$ such that

$$T(\mathbf{p}(t)) = \begin{pmatrix} \mathbf{p}(0) \\ \mathbf{p}(1) \\ \mathbf{p}(2) \end{pmatrix}$$

a. (3pts) Show that T is a linear transformation.

① $T(p+q) = \begin{pmatrix} (p+q)(0) \\ (p+q)(1) \\ (p+q)(2) \end{pmatrix} = \begin{pmatrix} p(0) \\ p(1) \\ p(2) \end{pmatrix} + \begin{pmatrix} q(0) \\ q(1) \\ q(2) \end{pmatrix} = T(p) + T(q)$

$(\forall p, q \in \mathbb{P}_2)$

② $T(cp) = \begin{pmatrix} (cp)(0) \\ (cp)(1) \\ (cp)(2) \end{pmatrix} = \begin{pmatrix} c\cdot p(0) \\ c\cdot p(1) \\ c\cdot p(2) \end{pmatrix} = c\cdot\begin{pmatrix} p(0) \\ p(1) \\ p(2) \end{pmatrix} = c\cdot T(p).$

$(\forall c \in \mathbb{R}, p \in \mathbb{P}_2)$

By ①, ②, T is a linear transformation

b. (6pts) Find $\ker T$. What is the dimension of $\ker T$? Conclude that im T is a 3-dimensional subspace of $\mathbb{R}^3$ so that im $T = \mathbb{R}^3$. (*Regard T as a linear transformation from $\mathbb{P}_2$ (3-dimensional vector space) to im T.*)

$\ker T = \{ p(t) \in \mathbb{P}_2 : T(p(t)) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \}$ by definition.

$= \{ p(t) \in \mathbb{P}_2 : p(0)=0, p(1)=0, p(2)=0 \}.$

~~Since p(0)=~~ Let $p(t) \in \mathbb{P}_2$ satisfy $p(0)=p(1)=p(2)=0$. Then, $c=0$ $(\because p(0)=0)$, $a+b+c=0$ $(\because p(1)=0)$, $4a+2b+c=0$ $(\because p(2)=0)$ $\Rightarrow a=b=c=0$.

$p(t) = at^2+bt+c$.

$= \{ 0\cdot t^2 + 0\cdot t + 0 \} = \{0\}.$

Hence, T is one-to-one. Therefore, im T has its dimension as same as the dimension of $\mathbb{P}_2$. Therefore, it has dimension 3. The only 3-dim'l subspace of a 3-dim'l space is itself. $\Rightarrow \text{im } T = \mathbb{R}^3$.

c. (6pts) Prove that T is one-to-one and onto. And then interpret the fact as following :

A polynomial of degree at most 2 is **uniquely** determined by three points $(0, \mathbf{p}(0))$, $(1, \mathbf{p}(1))$, and $(2, \mathbf{p}(2))$.

In problem b, we have shown that ~~T is~~ one-to-one. and since ~~T is onto~~, im $T = \mathbb{R}^3$, it is onto.

This implies that if $T(p(t)) = T(q(t))$, then $p(t) = q(t)$.

as well as that for every pair of three points $(0, p(0))$, $(1, p(1))$, $(2, p(2))$ (or three numbers} $p(0)$, $p(1)$, $p(2)$)

there exists a polynomial $p(t)$ s.t. $T(p(t)) = \begin{pmatrix} p(0) \\ p(1) \\ p(2) \end{pmatrix}$.

It means that $P(t)$ and $T(p(t))$ have one-to-one correspondence.

Hence, $p(t)$ is uniquely determined by three points $(0, p(0))$, $(1, p(1))$, $(2, p(2))$.