1. [1]Let V and W be vector spaces. Let $T : V \to W$ be a linear transformation. Suppose that $\dim V = n$.

a. (5pts) Prove that $\ker T$ is a subspace of V.

b. (5pts) Prove that $\text{im } T$ is a subspace of W.

[1]This problems is designed to give a proof for the generalized Rank Theorem, that is,

$$\dim \ker T + \dim \text{im} T = \dim V$$

c. (5pts) Let $\mathcal{B} = \{v_1, v_2, \cdots, v_m\}$ be a basis for $\ker T$ and extend the basis for $\ker T$ to a basis for V by adding $\{v_{m+1}, \cdots, v_n\}$. Show that

$$\begin{aligned} & T(a_1v_1 + \cdots + a_mv_m + a_{m+1}v_{m+1} + \cdots + a_nv_n) \\ = \ & a_{m+1}T(v_{m+1}) + \cdots + a_nT(v_n) \end{aligned}$$

and conclude that

$$\text{im } T = \text{Span}\{T(v_{m+1}), \cdots, T(v_n)\}$$

d. (5pts) Show that $\{T(v_{m+1}), \cdots, T(v_n)\}$ is linearly independent. Then, with the result of (c), conclude that

$$\mathcal{C} := \{T(v_{m+1}), \cdots, T(v_n)\}$$

is a basis for im T.

Therefore, $\dim \ker T = m$ and $\dim \text{im } T = n - (m+1) + 1 = n - m$ so that

$$\dim \ker T + \dim \text{im } T = n = \dim V$$

2. [2]Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space and W be a subspace of V. Given an orthogonal basis $\mathcal{B} = \{u_1, \cdots, u_m\}$ for W, recall that the formula of the orthogonal projection of $v \in V$ onto W is defined as

$$\frac{\langle u_1, v \rangle}{\langle u_1, u_1 \rangle} u_1 + \cdots + \frac{\langle u_m, v \rangle}{\langle u_m, u_m \rangle} u_m.$$

Let's denote the formula as $\text{proj}_{W,\mathcal{B}}(v)$.

a. (8pts) Show that $v - \text{proj}_{W,\mathcal{B}}(v)$ is orthogonal to $\text{proj}_{W,\mathcal{B}}(v)$ and also $v - \text{proj}_{W,\mathcal{B}}(v) \in W^{\perp}$. (Hint. Use the linearity property of an innder product $\langle \cdot, \cdot \rangle$ and the definition of *orthogonality*. In order to prove $v - \text{proj}_{W,\mathcal{B}} \in W^{\perp}$, you only need to show that $v - \text{proj}_{W,\mathcal{B}}$ is orthogonal to u_1, u_2, $\cdots$, u_m.)

[2]This problem is designed in order to prove that the formula for the orthogonal projection,

$$\frac{\langle u_1, v \rangle}{\langle u_1, u_1 \rangle} u_1 + \cdots + \frac{\langle u_m, v \rangle}{\langle u_m, u_m \rangle} u_m,$$

is independent of the choice of an orthogonal basis $\{u_1, u_2, \cdots, u_m\}$ for W.

b. (3pts) Let $\mathcal{C} = \{w_1, \cdots, w_m\}$ be another orthogonal basis for W.[3] Prove that[4]

$$\text{proj}_{W,\mathcal{B}}(v) - \text{proj}_{W,\mathcal{C}}(v) \in W^{\perp}.$$

c. (3pts) Assume that there is no nonzero vector v such that $v \in W$ and $v \in W^{\perp}$ at the same time, without a proof. Using this fact, prove that

$$\text{proj}_{W\mathcal{B}}(v) - \text{proj}_{W\mathcal{C}}(v) = 0$$

Therefore,

$$\text{proj}_{W\mathcal{B}}(v) = \text{proj}_{W\mathcal{C}}(v).$$

So, we can conclude that the formula of the orthogonal projection does not depend on the choice of an orthogonal basis.

[3]From a., we have $v - \text{proj}_{W,\mathcal{C}} \in W^{\perp}$.

[4]Hint. $W^{\perp}$ is a subspace of V (you can use this fact without a proof) so that $W^{\perp}$ is closed under addition and scalar multiplication.

3. (5pts) Find a general solution to

$$t^2y''(t) - 4ty'(t) + 6y(t) = 0.$$

4. (8pts) Given a differential equation in normal form

$$\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{f}(t)$$

the only information you have is a set of three solutions to the equation. They are

$$\begin{pmatrix} e^{-t} + e^{t} + e^{2t} \\ 2e^{t} - e^{2t} \end{pmatrix}, \quad \begin{pmatrix} -e^{-t} + e^{t} + e^{2t} \\ 2e^{t} - e^{2t} \end{pmatrix}, \quad \begin{pmatrix} e^{t} + e^{2t} \\ 2e^{t} - e^{2t} \end{pmatrix}.$$

Find $\mathbf{A}$ and $\mathbf{f}(t)$.

5. (15pts) Let A be a 3×3 matrix

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Compute e^{At} using generalized eigenvectors.[5]

[5]Hint. First, find the characteristic polynomial and then Nul $(A - \lambda I)^e$ where λ is an eigenvalue and e is the corresponding exponent. For $v \in$Nul$(A - \lambda I)^e$, you can compute $e^{At}v$ easily.

6. Let $C[-\pi, \pi]$ be the vector space of all continuous functions defined on $[-\pi, \pi]$. Let's define an inner product space $(C[-\pi, \pi], \langle \cdot, \cdot \rangle)$ by defining

$$\langle f(t), g(t) \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)g(t)dt.$$

a. (3pts) Show that $\sin mt$, $\sin nt$ are orthogonal if $m \neq n$.

b. (3pts) Show that $\sin mt$, $\cos nt$ are orthogonal.

c. (3pts) Show that $||\sin kt||^2 = ||\cos kt||^2 = 1$ for $k \neq 0$.

d. (9pts) Find the second-order Fourier Approximation of

$$e^t, \quad -\pi \leq t \leq \pi$$

7. (10pts) Let $\mathbf{A}$ and $\mathbf{B}$ be $n \times n$ constant matrices and $\mathbf{AB} = \mathbf{BA}$. Prove that[6]

$$e^{\mathbf{A}+\mathbf{B}} = e^{\mathbf{A}} e^{\mathbf{B}}.$$

[6]Hint.

$$(x+y)^n = \frac{n!}{n!}x^n + \frac{n!}{(n-1)!}x^{n-1}y + \frac{n!}{2!(n-2)!}x^{n-2}y^2 + \cdots + \frac{n!}{k!(n-k)!}x^{n-k}y^k + \cdots + \frac{n!}{n!}y^n.$$