1 Ortho...

1.1 Orthogonal set and Orthogonal basis

A set of vectors is said to be an **orthogonal set** if each pair of distinct vectors is orthogonal. If the set is a basis, then we call the set as an **orthogonal basis**.

Intuitively, since those vectors are mutually perpendicular, they are linearly independent. In fact, it reall is. Why?

When we discussed about $\mathcal{B}$-coordinates, one important issue was how to find the coefficients $c_1, \cdots, c_p$ for

$$\mathbf{y} = c_1\mathbf{u}_1 + \cdots c_p\mathbf{u}_p.$$

where $\mathcal{B} = \{\mathbf{u}_1, \cdots, \mathbf{u}_p\}$. It is much easier when $\mathcal{B}$ is an *orthogonal basis*. $c_j = \frac{\mathbf{y}\cdot\mathbf{u}_j}{\mathbf{u}_j\cdot\mathbf{u}_j}$. This is the reason why we are interested in an *orthogonal basis*.

Orthonormal. An **orthonormal set** or **basis** is an **orthogonal set** or basis of which vectors are all unit vectors.

1.2 Geometric meaning of the standard dot product

Given any two vectors, we naturally have an angle, say θ, between those two vectors. When $\theta = 0$, two vectors have the same direction. In fact, we have more informative formula

$$u \cdot v = ||u|| \cdot ||v|| \cos\theta.$$

There are a few things to note.

a. If v is a unit vector, then $u \cdot v$ is the length of the orthogonal projection of u onto v.

b. In general, $u \cdot v/||v||$ is the length of the orthogonal projection of u onto v.

c. $u \cdot v \leq ||u|| \cdot ||v||$.

1.3 Orthogonal Projection $\hat{\mathbf{y}}$

2 types : onto $\mathbf{u}$ vs. onto W (a subspace of $\mathbb{R}^n$)

1.3.1 Onto u

$$\hat{\mathbf{y}} = \text{proj}_L \mathbf{y} = \frac{\mathbf{y} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}} \mathbf{u}$$

where L is the subspace spanned by $\mathbf{u}$.

1.3.2 Onto W

$$\hat{\mathbf{y}} = \text{proj}_W \mathbf{y} = \frac{\mathbf{y} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1} \mathbf{u}_1 + \cdots + \frac{\mathbf{y} \cdot \mathbf{u}_p}{\mathbf{u}_p \cdot \mathbf{u}_p} \mathbf{u}_p$$

where $\{\mathbf{u}_1, \cdots, \mathbf{u}_p\}$ is an orthogonal basis of W.

2 The Gram-Schmidt Process

Given a basis $\{\mathbf{x}_1, \cdots, \mathbf{x}_p\}$ for a nonzero subspace W of $\mathbb{R}^n$, define

$$\begin{aligned}
\mathbf{v}_1 &= \mathbf{x}_1 \\
\mathbf{v}_2 &= \mathbf{x}_2 - \tfrac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \mathbf{v}_1 \\
\mathbf{v}_3 &= \mathbf{x}_3 - \tfrac{\mathbf{x}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \mathbf{v}_1 - \tfrac{\mathbf{x}_3 \cdots \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2} \mathbf{v}_2 \\
&\vdots
\end{aligned}$$

Then $\{\mathbf{v}_1, \cdots, \mathbf{v}_p\}$ is an orthogonal basis for W. In addition

$$\text{Span}\{\mathbf{v}_1, \cdots, \mathbf{v}_k\} = \text{Span}\{\mathbf{x}_1, \cdots, \mathbf{x}_k\} \text{ for all } 1 \leq k \leq p$$

1. Show that $\{\mathbf{u}_1, \mathbf{u}_2\}$ or $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthogonal basis for $\mathbb{R}^2$ or $\mathbb{R}^3$, respectively. Then express $\mathbf{x}$ as a linear combination of the $\mathbf{u}$'s.

   a. $\mathbf{u}_1 = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$, $\mathbf{u}_2 = \begin{pmatrix} 6 \\ 4 \end{pmatrix}$, and $\mathbf{x} = \begin{pmatrix} 9 \\ -7 \end{pmatrix}$.

   b. $\mathbf{u}_1 = \begin{pmatrix} 3 \\ -3 \\ 0 \end{pmatrix}$, $\mathbf{u}_2 = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$, $\mathbf{u}_3 = \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}$, and $\mathbf{x} = \begin{pmatrix} 5 \\ -3 \\ 1 \end{pmatrix}$.

2. Let $\mathbf{y} = \begin{pmatrix} -3 \\ 9 \end{pmatrix}$ and $\mathbf{u} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Compute the distance from $\mathbf{y}$ to the line through $\mathbf{u}$ and the origion.

3. Mark each statement True or False. Justify your answer. (All vectors are in $\mathbb{R}^n$.)

   a. Not every linearly independent set in $\mathbb{R}^n$ is an orthogonal set.

   b. A matrix with orthonormal columns is an orthogonal matrix.

   c. If L is a line through $\mathbf{0}$ and if $\hat{\mathbf{y}}$ is the orthogonal projection of $\mathbf{y}$ onto L, then $||\hat{\mathbf{y}}||$ gives the distance from $\mathbf{y}$ to L.

   d. Not every orthogonal set in $\mathbb{R}^n$ is linearly independent.

   e. If the columns of an $m \times n$ matrix A are orthonormal, then the linear mapping $\mathbf{x} \mapsto A\mathbf{x}$ preserves length.

   f. An orthognoal matrix is invertible.

4. Let U be an $m \times n$ matrix with orthonormal columns, and let $\mathbf{x}$ and $\mathbf{y}$ be in $\mathbb{R}^n$. Show that

   a. $||U\mathbf{x}|| = ||\mathbf{x}||$.

   b. $(U\mathbf{x}) \cdot (U\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$.

   c. $(U\mathbf{x}) \cdot (U\mathbf{y}) = 0$ if and only if $\mathbf{x} \cdot \mathbf{y} = 0$.

5. Suppose that W is a subspace of $\mathbb{R}^n$ spanned by n nonzero orthogonal vectors. Explain why $W = \mathbb{R}^n$.

6. Let U be an $n \times n$ orthogonal matrix. Show that the rows of U form an orthonormal basis of $\mathbb{R}^n$.

7. Let U and V be $n \times n$ orthogonal matrices. Explain why UV is an orthogonal matrix.

8. Find the closest point to $\mathbf{y}$ in the subspace W spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$.

$$\mathbf{y} = \begin{pmatrix} 3 \\ 1 \\ 5 \\ 1 \end{pmatrix}, \quad \mathbf{v}_1 = \begin{pmatrix} 3 \\ 1 \\ -1 \\ 1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$$

9. Let W be a subspace of $\mathbb{R}^n$ with an orthogonal basis $\{\mathbf{w}_1, \cdots, \mathbf{w}_p\}$, and let $\{\mathbf{v}_1, \cdots, \mathbf{v}_q\}$ be an orthogonal basis for $W^\perp$.

   a. Explain why $\{\mathbf{w}_1, \cdots, \mathbf{w}_p, \mathbf{v}_1, \cdots, \mathbf{v}_1\}$ is an orthogonal set.

   b. Explain why the set in part (a) spans $\mathbb{R}^n$.

   c. Show that $\dim W + \dim W^\perp = n$.

10. Mark each statement True or False. Justify your answer. (All vectors and subspaces are in $\mathbb{R}^n$.)

    a. If $\mathbf{z}$ is orthogonal to $\mathbf{u}_1$ and $\mathbf{u}_2$ and if $W = \mathrm{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$, then $\mathbf{z}$ must be in $W^\perp$.

    b. For each $\mathbf{y}$ and each subspace W, the vector $\mathbf{y} - \mathrm{proj}_W \mathbf{y}$ is orthogonal to W.

    c. The orthogonal projection $\hat{\mathbf{y}}$ of $\mathbf{y}$ onto a subspace W can sometimes depend on the orthogonal basis for W used to compute $\hat{\mathbf{y}}$.

    d. If W is a subspace of $\mathbb{R}^n$ and if $\mathbf{v}$ is in both W and $W^\perp$, then $\mathbf{v}$ must be the zero vector.

    e. If $\mathbf{y} = \mathbf{z}_1 + \mathbf{z}_2$, where $\mathbf{z}_1$ is in a subspace W and $\mathbf{z}_2$ is in $W^\perp$, then $\mathbf{z}_1$ must be the orthogonal projection of $\mathbf{y}$ onto W.