1 Eigen...

1.1 Eigenspace corresponding to λ

There is no notation for the eigenspace corresponding to λ in the textbook. I would like to denote *the eigenspace corresponding to* λ as E_λ. The defintion of E_λ is

$$E_\lambda = \text{Nul }(A - \lambda I).$$

1.2 The characteristic polynomial

Let's first think about an example.

Let A be the 3×3 matrix

$$\begin{pmatrix} 2 & 0 & 3 \\ 7 & -1 & 3 \\ 1 & 0 & 4 \end{pmatrix}$$

Find all eigenvalues of the matrix.

We need to find λ satisfying

$$\det(A - \lambda I) = 0.$$

Now, let's see what $A - \lambda I$ is.

$$A - \lambda I = \begin{pmatrix} 2-\lambda & 0 & 3 \\ 7 & -1-\lambda & 3 \\ 1 & 0 & 4-\lambda \end{pmatrix}$$

If we take the second column to calculate the determinant, then $\det(A - \lambda I)$ is

$$(-1-\lambda)\det\begin{pmatrix} 2-\lambda & 3 \\ 1 & 4-\lambda \end{pmatrix}.$$

So, $\det(A - \lambda I) = (-1-\lambda)((2-\lambda)(4-\lambda) - 3)$.

Therefore, basically, find an eigenvalue is solving a polynomial equation which is in this case

$$(-1-\lambda)((2-\lambda)(4-\lambda) - 3) = 0.$$

Solving this equation, we get $(-1-\lambda)(\lambda^2 - 6\lambda + 5) = 0$ and then $-(1+\lambda)(\lambda-5)(\lambda-1) = 0$. Hence, we have three eigenvalues $\lambda_1 = -1$, $\lambda_2 = 1$, $\lambda_3 = 5$.

As we can see from the above example, solving

$$\det(A - \lambda I) = 0$$

is always solving a polynomial equation. In other words, $\det(A - \lambda I)$ is a polynomial of λ. This polynomial is called **the characteristic polynomial of** A, denoted by χ_A. Moreover, to be precise, $\det(A - \lambda I)$ is a polynomial of degree **exactly** n when A is an $n \times n$ matrix.

1.3 Theorem 2 : Property of distinct eigenvalues

Think about the above example. There are 3 different eigenvalues λ_1, λ_2, and λ_3. Then, we have three eigenspaces :

$$E_{\lambda_1},\ E_{\lambda_2},\ E_{\lambda_3}$$

Now, let's go back to the example.

By definition, $E_{\lambda_1} = \text{Nul}\,(A+I)$. So, explicitly,

$$E_{\lambda_1} = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} : \begin{pmatrix} 3 & 0 & 3 \\ 7 & 0 & 3 \\ 1 & 0 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

Solving the matrix equation, we get $x_1 = 0$, $x_3 = 0$, and x_2 is free. Hence,

$$E_{\lambda_1} = \left\{ \begin{pmatrix} 0 \\ x_2 \\ 0 \end{pmatrix} : x_2 \in \mathbb{R} \right\} = \text{Span}\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

In a similar way, we can figure out that

$$E_{\lambda_2} = \text{Span}\left\{ \begin{pmatrix} 3 \\ 9 \\ -1 \end{pmatrix} \right\}, \quad E_{\lambda_3} = \text{Span}\left\{ \begin{pmatrix} 3 \\ 5 \\ 3 \end{pmatrix} \right\}.$$

If we choose three vectors, one from each E_λ's, then those three vectors are linearly independent. This is not only for this specific example. Generally, we have

> If v_1, v_2, $\cdots$, v_r are eigenvectors that correspond to distinct eigenvectors λ_1, λ_2, $\cdots$, λ_r of an $n \times n$ matrix A, then the set $\{v_1, v_2, \cdots, v_r\}$ is a linearly independent set.

Also, in this specific example, we have three different eigenvalues for a 3×3 matrix. As we have seen in the class, there might not be enough eigenvalues. However, if it is the case, that is,

> If there are n different eigenvalues λ_1, λ_2, $\cdots$, λ_n for an $n \times n$ matrix A, then $\{v_1, v_2, \cdots, v_n\}$ is a basis of $\mathbb{R}^n$ where v_i is an eigenvector corresponding to λ_i.

2 Diagonalization

Before we study The Diagonalization Theorem, let's see three examples below.

$$A = \begin{pmatrix} 2 & 0 & 3 \\ 7 & -1 & 3 \\ 1 & 0 & 4 \end{pmatrix} \qquad B = \begin{pmatrix} -2 & 6 & -6 \\ -5 & 11 & -10 \\ -3 & 6 & -5 \end{pmatrix} \qquad C = \begin{pmatrix} 0 & 4 & -6 \\ 4 & 3 & -3 \\ 3 & 0 & 1 \end{pmatrix}$$

$$\chi_A = -(\lambda+1)(\lambda-1)(\lambda-5) \qquad \chi_B = -(\lambda-1)^2(\lambda-2) \qquad \chi_C = -(\lambda-1)^2(\lambda-2)$$

1) A has eigenvalues -1, 1, and 5. So, there are three eigenvectors that form a basis of $\mathbb{R}^3$. Meanwhile,

2) B has eigenvalues 1 and 2. Surely, there are two linearly independent eigenvectors. In fact, it turns out to be that there are **three linearly independent eigenvectors** which end up with forming a basis of $\mathbb{R}^3$. Also,

3) C has eigenvalues 1 and 2. Surely, there are two linearly independent eigenvectors. In fact, it turns out to be that there are **NO three linearly independent eigenvectors** which are not able to form a basis of $\mathbb{R}^3$. To be precise,

$$v_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad v_2 = \begin{pmatrix} 3 \\ 9 \\ -1 \end{pmatrix} \quad v_3 = \begin{pmatrix} 3 \\ 5 \\ 3 \end{pmatrix} \qquad Av_1 = -v_1 \quad Av_2 = v_2 \quad Av_3 = 5v_3$$

$$v_1 = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} \quad v_2 = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \quad v_2' = \begin{pmatrix} 3 \\ 5 \\ 3 \end{pmatrix} \qquad Bv_1 = v_1 \quad Bv_2 = v_2 \quad Bv_3 = 2v_3$$

$$v_1 = \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix} \quad v_2 = \begin{pmatrix} 1 \\ 5 \\ 3 \end{pmatrix} \quad v_? = \begin{pmatrix} ? \\ ? \\ ? \end{pmatrix} \qquad Cv_1 = v_1 \quad Cv_2 = 2v_2 \quad Cv_? = ?v_?$$

Attaching v_1, v_2, v_3 side by side, we get a 3×3 matrix, denoted by $P = \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix}$. From the example, we have

$$A \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix} = \begin{pmatrix} -v_1 & v_2 & 5v_3 \end{pmatrix}.$$

Multiplying P on the left of each sides, we get

$$P^{-1}AP = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{pmatrix}.$$

Consequently, we get a diagonal matrix. This happens since we have a basis of $\mathbb{R}^3$ consisting of eigenvectors.

2.1 Theorem 5 : The Diagonaliztion Theorem

An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.

1. Let A be an $n \times n$ matrix, and suppose A has n real eigenvalues, $\lambda_1, \lambda_2, \cdots, \lambda_n$, repeated according to multiplicities, so that
$$\det(A - \lambda I) = (\lambda_1 - \lambda)(\lambda_2 - \lambda)\cdots(\lambda_n - \lambda).$$
Explain why $\det A$ is the product of the n eigenvalues of A.

2. Show that A and A^T have the same characteristic polynomial.

3. Mark each statement True or False.
   a. Every vector v in E_λ is an eigenvector.
   b. Every 3×3 matrix A is not diagonalizable unless A has three distinct eigenvalues.
   c. Every $n \times n$ matrix A which has n distinct eigenvalues is always diagonalizable.
   d. The determinant of A is the product of the diagonal entries in A.
   e. An elementary row operation on A does not change the determinant.

4. Let $A = PDP^{-1}$ and compute A^T.
$$P = \begin{pmatrix} 5 & 7 \\ 2 & 3 \end{pmatrix}, \quad D = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

5. Let $A = PDP^{-1}$.
$$\begin{pmatrix} 3 & 0 & 0 \\ -3 & 4 & 9 \\ 0 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 3 & 0 & -1 \\ 0 & 1 & -3 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ -3 & 1 & 9 \\ -1 & 0 & 3 \end{pmatrix}$$
Find the eigenvalues of A and a basis for each eigenspace.

6. Let A, B, P, and D are $n \times n$ matrices. Mark each statement True or False. Justify each answer.

   a. If $\mathbb{R}^n$ has a basis of eigenvectors of A, then A is diagonalizable.

   b. A is diagonalizable if and only if A has n eigenvalues, counting multiplicities.

   c. If A is diagonalizable, then A has n distinct eigenvalues.

   d. If A is diagonalizable, then A is invertible.

   e. If A is invertible, then A is diagonalizable.

7. A is a 5×5 matrix with with two eigenvalues. One eigenspace is three-dimensional, and the other eigenspace is two-dimensional. Is A diagonalizable? Why?

8. Show that if A is both diagonalizable and invertible, then so is A^{-1}.

9. Construct a nonzero 2×2 matrix that is invertible but not diagonalizable.