# 1 Change of Basis

## 1.1 Change of Basis in $\mathbb{R}^n$

Let $\mathbf{b}_1 = \begin{pmatrix}-9\\1\end{pmatrix}$, $\mathbf{b}_2 = \begin{pmatrix}-5\\-1\end{pmatrix}$, $\mathbf{c}_1 = \begin{pmatrix}1\\-4\end{pmatrix}$, $\mathbf{c}_2 = \begin{pmatrix}3\\-5\end{pmatrix}$. Find $\underset{C\leftarrow B}{P}$.

$$\begin{pmatrix}1&3&-9&-5\\-4&-5&1&-1\end{pmatrix} \sim \begin{pmatrix}1&3&-9&-5\\0&7&-35&-21\end{pmatrix} \sim \begin{pmatrix}1&3&-9&-5\\0&1&-5&-3\end{pmatrix} \sim \begin{pmatrix}1&0&6&4\\0&1&-5&-3\end{pmatrix}.$$

Hence, $\underset{C\leftarrow B}{P} = \begin{pmatrix}6&4\\-5&-3\end{pmatrix}$

## 1.2 Change of Basis in $\mathbb{P}_n$

Let $\mathcal{B} = \{1-3t^2, 2+t-5t^2, 1+2t\}$ in $\mathbb{P}_2$. Find the change-of-coordinates matrix from the basis $\mathcal{B}$ to the standard basis. Then wrtie $t^2$ as a linear combination of the polynomials in $\mathcal{B}$.

$1-3t^2 = 1\cdot 1 + 0\cdot t + (-3)t^2 \Rightarrow$ first column is $\begin{pmatrix}1\\0\\-3\end{pmatrix}$

$2+t-5t^2 = 2\cdot 1 + 1\cdot t + (-5)t^2 \Rightarrow$ second column $= \begin{pmatrix}2\\1\\-5\end{pmatrix}$

$1+2t = 1\cdot 1 + 2\cdot t + 0\cdot t^2 \Rightarrow$ third column $= \begin{pmatrix}1\\2\\0\end{pmatrix}$

$\therefore \underset{\mathcal{E}\leftarrow\mathcal{B}}{P} = \begin{pmatrix}1&2&1\\0&1&2\\-3&-5&0\end{pmatrix}$ where $\mathcal{E} = \{1, t, t^2\}$ : the standard basis.

We need to read $t^2$ using $\mathcal{B}$.

$[t^2]_{\mathcal{B}} = \underset{\mathcal{B}\leftarrow\mathcal{E}}{P}[t^2]_{\mathcal{E}}$

$= \left(\underset{\mathcal{E}\leftarrow\mathcal{B}}{P}\right)^{-1}\cdot\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}10&-5&3\\-6&3&-2\\3&-1&1\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}3\\-2\\1\end{pmatrix}$

# 2 Eigenvalues and Eigenvectors

From now on, we will only discuss about square matrices. Let $A$ be an $n\times n$ matrix. Assume that there exists a real number $\lambda$ such that

$$\det(A-\lambda I) = 0.$$

We call such real number as **an eigenvalue of** $A$. If you remind the Invertible Matrix Theorem, it implies that Nul $(A-\lambda I)$ is not the zero space. Every nonzero vector $v \in$ Nul $(A-\lambda I)$ is called **an eigenvector**.

1. If $A$ is a $7 \times 5$ matrix, what is the largest possible rank of $A$? If $A$ is a $5 \times 7$ matrix, what is the largest possible rank of $A$? Explain your answers.

5 and 5. From 2.a, we know that rank $A$ is dim Row $A$ = dim Col $A$. $7\times5$: Row $A \subseteq \mathbb{R}^5$, $5\times7$: Col $A \subseteq \mathbb{R}^5$. Hence, their maximal dimensions are 5.

Ex. $\begin{pmatrix} I_5 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} I_5 & 0 & 0 \end{pmatrix}$

2. Mark each statement True or False. Justify your answer.

a. The dimensions of the row space and the column space of $A$ are the same, even if $A$ is not square.

True. Because of 2.c, Row operations do not change linear dependence relation among the rows of A

b. If $B$ is the reduced echelon form of $A$, and if $B$ has exactly three nonzero rows, then the first three rows of $A$ form a basis for Row $A$.

True. Because of 2.C, Row operations preserve the lin. dep relations among the rows A.

c. Row operations preserve the linear dependence relations among the rows of $A$.

True. In class, we did.

d. If $A$ an $B$ are row equivalent, then their row spaces are the same.

True. Row operations do not change the space.

3. Suppose $A$ is $m \times n$ and $\mathbf{b}$ is in $\mathbb{R}^m$. What has to be true about the two numbers rank $(A\ \mathbf{b})$ and rank $A$ in order for the equation $A\mathbf{x} = \mathbf{b}$ to be consistent?

$$\operatorname{rank}(A\ \mathbf{b}) = \operatorname{rank} A.$$

4. a. Is $\lambda = 2$ an eigenvalue of $\begin{pmatrix} 3 & 2 \\ 3 & 8 \end{pmatrix}$?

$$\det\left(\begin{pmatrix} 3 & 2 \\ 3 & 8 \end{pmatrix} - \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\right) = \det\left(\begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}\right) = 0 \quad \therefore \lambda = 2 \text{ is an eigenvalue of } \begin{pmatrix} 3 & 2 \\ 3 & 8 \end{pmatrix}.$$

b. Is $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ an eigenvector of $\begin{pmatrix} 5 & 2 \\ 3 & 6 \end{pmatrix}$? If so, find the eigenvalue.

$$\begin{pmatrix} 5 & 2 \\ 3 & 6 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ 3 \end{pmatrix} = 3 \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$ $\therefore$ It is true, and the corresponding eigenvalue is 3.

c. Is $\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$ an eigenvector of $\begin{pmatrix} 3 & 6 & 7 \\ 3 & 2 & 7 \\ 5 & 6 & 4 \end{pmatrix}$? If so, find the eigenvalue.

$$\begin{pmatrix} 3 & 6 & 7 \\ 3 & 2 & 7 \\ 5 & 6 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 13 \\ 1 \end{pmatrix} \neq \text{not a multiple of } \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}.$$ It is NOT true.

5. Let $\lambda$ be an eigenvalue of an invertible matrix $A$. Show that $\lambda^{-1}$ is an eigenvalue of $A^{-1}$.

① $(A^{-1}-\lambda^{-1}I)\cdot\lambda A = (\lambda I - A)$

② $Av=\lambda v$ then $A^{-1}Av = \lambda A^{-1}v$ and then $\lambda^{-1}Iv = A^{-1}v$ so that $A^{-1}v=\lambda^{-1}v$. Therefore, $\lambda^{-1}$ is an eigenvalue of $A^{-1}$

Therefore, $\det(A^{-1}-\lambda^{-1}I) = \det((\lambda I - A)\lambda^{-1}A^{-1}) = \det(\lambda I - A)\cdot\det(\lambda^{-1}A^{-1}) = 0$. Therefore $\lambda^{-1}$ is an eigenvalue of $A^{-1}$. $\blacksquare$

6. Show that if $A^2$ is the zero matrix, then the only eigenvalue of $A$ is 0.

Suppose that $Av=\lambda v$ for some scalar $\lambda$ and $v$: vector.
Multiplying $A$ on left sides of each sides,
$0 = 0\cdot v = A^2v = \lambda Av = \lambda(\lambda v) = \lambda^2 v$. Hence, $\lambda^2=0 \Rightarrow \lambda=0$. $\blacksquare$

7. Show that $\lambda$ is an eigenvalue of $A$ if and only if $\lambda$ is an eigenvalue of $A^T$.

$\det(A-\lambda I) = \det((A-\lambda I)^T) = \det(A^T-\lambda I)$ $\blacksquare$

8. Let $A$ be an $n\times n$ matrix. Mark each statement True or False. Justify your answer.

a. If $A\mathbf{x} = \lambda\mathbf{x}$ for some vector $\mathbf{x}$, then $\lambda$ is an eigenvalue of $A$.

False. $x$ should not be the zero vector.

b. A number $c$ is an eigenvalue of $A$ if and only if the equation $(A-cI)\mathbf{x} = \mathbf{0}$ has a nontrivial solution.

True. $B\vec{x}=\vec{0}$ has a nontrivial sol'n if and only if $\det B = 0$.

c. If $v_1$ and $v_2$ are linearly independent eigenvectors, then they correspond to distinct eigenvalues.

False. Counterexample. $\begin{pmatrix}1&0\\0&1\end{pmatrix}$ : $\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}$ are eigenvectors and they are linearly independent, but have the same eigenvalue.

9. Construct an example of a $2\times 2$ matrix with only one distinct eigenvalue.

$\begin{pmatrix}1&0\\0&1\end{pmatrix}$ has only one distinct eigenvalue $\lambda=1$.