1 Change of Basis

1.1 Change of Basis in $\mathbb{R}^n$

Let $\mathbf{b}_1 = \begin{pmatrix} -9 \\ 1 \end{pmatrix}$, $\mathbf{b}_2 = \begin{pmatrix} -5 \\ -1 \end{pmatrix}$, $\mathbf{c}_1 = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$, $\mathbf{c}_2 = \begin{pmatrix} 3 \\ -5 \end{pmatrix}$. Find $\underset{C \leftarrow B}{P}$.

1.2 Change of Basis in $\mathbb{P}_n$

Let $\mathcal{B} = \{1 - 3t^2, 2 + t - 5t^2, 1 + 2t\}$ in $\mathbb{P}_2$. Find the change-of-coordinates matrix from the basis $\mathcal{B}$ to the standard basis. Then wrtie t^2 as a linear combination of the polynomials in $\mathcal{B}$.

2 Eigenvalues and Eigenvectors

From now on, we will only discuss about square matrices. Let A be an $n \times n$ matrix. Assume that there exists a real number λ such that

$$\det(A - \lambda I) = 0.$$

We call such real number as **an eigenvalue of** A. If you remind the Invertible Matrix Theorem, it implies that Nul $(A - \lambda I)$ is not the zero space. Every nonzero vector $v \in$ Nul $(A - \lambda I)$ is called **an eigenvector**.

1. If A is a 7×5 matrix, what is the largest possible rank of A? If A is a 5×7 matrix, what is the largest possible rank of A? Explain your answers.

2. Mark each statement True or False. Justify your answer.

   a. The dimensions of the row space and the column space of A are the same, even if A is not square.

   b. If B is the reduced echelon form of A, and if B has exactly three nonzero rows, then the first three rows of A form a basis for Row A.

   c. Row operations preserve the linear dependence relations among the rows of A.

   d. If A an B are row equivalent, then their row spaces are the same.

3. Suppose A is $m \times n$ and $\mathbf{b}$ is in $\mathbb{R}^m$. What has to be true about the two numbers rank $(A\ \mathbf{b})$ and rank A in order for the equation $A\mathbf{x} = \mathbf{b}$ to be consistent?

4. a. Is $\lambda = 2$ an eigenvalue of $\begin{pmatrix} 3 & 2 \\ 3 & 8 \end{pmatrix}$?

   b. Is $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ an eigenvector of $\begin{pmatrix} 5 & 2 \\ 3 & 6 \end{pmatrix}$? If so, find the eigenvalue.

   c. Is $\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$ an eigenvector of $\begin{pmatrix} 3 & 6 & 7 \\ 3 & 2 & 7 \\ 5 & 6 & 4 \end{pmatrix}$? If so, find the eigenvalue.

5. Let λ be an eigenvalue of an invertible matrix A. Show that λ^{-1} is an eigenvalue of A^{-1}.

6. Show that if A^2 is the zero matrix, then the only eigenvalue of A is 0.

7. Show that λ is an eigenvalue of A if and only if λ is an eigenvalue of A^T.

8. Let A be an $n \times n$ matrix. Mark each statement True or False. Justify your answer.

   a. If $A\mathbf{x} = \lambda\mathbf{x}$ for some vector $\mathbf{x}$, then λ is an eigenvalue of A.

   b. A number c is an eigenvalue of A if and only if the equation $(A - cI)\mathbf{x} = \mathbf{0}$ has a nontrivial solution.

   c. If v_1 and v_2 are linearly independent eigenvectors, then they correspond to distinct eigenvalues.

9. Construct an example of a 2×2 matrix with only one distinct eigenvalue.