1 Vector space $\mathbb{R}^n$

A vector space $\mathbb{R}^n$ is the set of all possible *ordered pairs* of n real numbers. So,

$$\mathbb{R}^n = \{(a_1, a_2, \cdots, a_n) : a_1,\ a_2,\ \cdots,\ a_n \in \mathbb{R}\}.$$

We abuse the notation $(a_1, a_2, \cdots, a_n)$ instead of $\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$ sometimes.

The meaning of A being a subset of B is that for every element a in A it is included in B. So, $\mathbb{R}^2$ is not a subset of $\mathbb{R}^3$. However, $\mathbb{R}^2 \times \{0\}$ is a subset of $\mathbb{R}^3$. Since what we mean by $\mathbb{R}^2 \times \{0\}$ is

$$\mathbb{R}^2 \times \{0\} = \{(a, b, 0) : a,\ b \in \mathbb{R}\}.$$

1.1 A subspace of $\mathbb{R}^n$

What does it mean by that H is a subspace of $\mathbb{R}^n$?

First of all, of course, we require H to be a subset of $\mathbb{R}^n$. Hence, for instance, $\mathbb{R}^2$ can never be **a subspace** of $\mathbb{R}^3$. However, $\mathbb{R}^2 \times \{0\}$ can be a subspace of $\mathbb{R}^3$ and it is indeed. Now, for a subset H to be a subspace we require three conditions : 1) H has the zero vector 2) H is closed under addition 3) H is closed under scalar multiplication.

1) In the case of $\mathbb{R}^2$, the zero vector of $\mathbb{R}^2$ is $(0, 0)$. Hence, for H to be a subspace of $\mathbb{R}^2$, H should have $(0, 0)$. If the case is $\mathbb{R}^3$, then H should have $(0, 0, 0)$. For the next two conditions, let's think about the meaning of 'closed'.

2, 3) Example. The set of all odd numbers $\mathbb{Z}_{odd}$ and the set of all even numbers $\mathbb{Z}_{even}$.

$\mathbb{Z}_{odd}$ is **NOT** closed under addition, meanwhile $\mathbb{Z}_{even}$ is closed under addition. Why?

In this sense... for H to be a subspace of $\mathbb{R}^n$, we require addition of any two vectors in H should be inside of H again. Also, we require the multiplication of any vector in H with any real number belongs to H. If these conditions are satisfied H is a subspace of $\mathbb{R}^n$.

1.2 A vector space $\mathbb{P}_n$

The set of all polynomials of degree at most n will be denoted by $\mathbb{P}_n$. Examples of a **polynomial of degree** 5 could be

$$\begin{aligned}&t^5 - 2t^4 + t^2 - 7t + 1\\&t^5 - t^2 + t\\&t^5 + 1\\&2t^5\end{aligned}$$

So, these polynomials below is also inside of $\mathbb{P}_5$:

$$\begin{aligned}&t^4 - t^2 + 7\\&t^3 + 1\\&t^2 + 2t + 1\\&1\end{aligned}$$

Here, 1 (one) is **a** polynomial of degree 0. $\mathbb{P}_n$ is also a vector space. What *a vector* means in this specific vector space is *a polynomial of degree at most n*. **The zero vector** of the vector space $\mathbb{P}_n$ is $0 \cdot 1 + 0 \cdot t + \cdots + 0 \cdot t^n$, or shortly 0.

Please note that ***we do not define the multiplication between two vectors***. So, when we think of $\mathbb{P}_n$ **as a vector space**, we do **NOT** define $t \times t$ or $(t + 1) \times (t^4 - t^3 + 1)$.

> We only have the *addition of two vectors (in other words, two polynomials)* and the *scalar multiplication of one vector (in other words, one polynomial) and one scalar (in other words, a real number).*

The addition of two vectors

$$\begin{aligned}t^5 + t^3 - 2t + 1\\t^3 + 5t^2 + 3t\end{aligned}$$

is $t^5 + 2t^3 + 5t^2 + t + 1$.

The scalar multiplication of a vector

$$t^4 - t^3 + 2t + 8$$

and a scalar -7 is $-7t^4 + 7t^3 - 14t - 56$.

1.2.1 A subspace of $\mathbb{P}_n$

Similarly as before, for a set H to be a subspace of $\mathbb{P}_n$, we first require H to be a subset of $\mathbb{P}_n$. And then, it is the same as before (the case of $\mathbb{R}^n$), we require three conditions. 1) H should have the zero vector 2) H is closed under addition 3) H is closed under scalar multiplication.

Is

$$S = \{\mathbf{p}(t) \in \mathbb{P}_4 : \mathbf{p}(0) = 0\}$$

a subspace of $\mathbb{P}_4$?

Is

$$S = \{\mathbf{p}(t) \in \mathbb{P}_4 : \mathbf{p}(0) = 1\}$$

a subspace of $\mathbb{P}_4$?

2 Col A and Nul A

Let A be an $m \times n$ matrix.

Col A is the **column space** of a matrix A.

Given a matrix A, it is **NOT** a matrix or **NOT** a vector but a set of specific vectors.

$$\begin{aligned}\text{Col } A &= \{A\mathbf{u} : \mathbf{u} \in \mathbb{R}^n\} \\ &= \text{Span}\{\text{column vectors of } A\} \\ &= \{\text{all linear combinations of column vectors of } A\}\end{aligned}$$

For example, given a matrix

$$A = \begin{pmatrix} 3 & 7 & 0 \\ 1 & -1 & 0 \\ 2 & 2 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

the column space Col A contains vectors

$$\begin{pmatrix} 10 \\ 0 \\ 7 \\ 0 \end{pmatrix} = 1\begin{pmatrix} 3 \\ 1 \\ 2 \\ 0 \end{pmatrix} + 1\begin{pmatrix} 7 \\ -1 \\ 2 \\ 0 \end{pmatrix} + 1\begin{pmatrix} 0 \\ 0 \\ 3 \\ 0 \end{pmatrix} \quad \text{and also} \quad \begin{pmatrix} -8 \\ 4 \\ 9 \\ 0 \end{pmatrix} = 2\begin{pmatrix} 3 \\ 1 \\ 2 \\ 0 \end{pmatrix} + (-2)\begin{pmatrix} 7 \\ -1 \\ 2 \\ 0 \end{pmatrix} + 3\begin{pmatrix} 0 \\ 0 \\ 3 \\ 0 \end{pmatrix}$$

Obviously, Col A contains the zero vector in $\mathbb{R}^4$, $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = 0\begin{pmatrix} 3 \\ 1 \\ 2 \\ 0 \end{pmatrix} + 0\begin{pmatrix} 7 \\ -1 \\ 2 \\ 0 \end{pmatrix} + 0\begin{pmatrix} 0 \\ 0 \\ 3 \\ 0 \end{pmatrix}$.

Here, please note that, by the definition of the multiplcation of an $m \times n$ matrix and a vector in $\mathbb{R}^n$

$$\begin{pmatrix} 10 \\ 0 \\ 7 \\ 0 \end{pmatrix} = 1\begin{pmatrix} 3 \\ 1 \\ 2 \\ 0 \end{pmatrix} + 1\begin{pmatrix} 7 \\ -1 \\ 2 \\ 0 \end{pmatrix} + 1\begin{pmatrix} 0 \\ 0 \\ 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 & 7 & 0 \\ 1 & -1 & 0 \\ 2 & 2 & 3 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Hence, we can say that Col $A = \{A\mathbf{u} : \mathbf{u} \in \mathbb{R}^n\}$ in general.

Conventionally, we use notations $e_1, e_2, \cdots, e_n$ for[1]

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix},\ e_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \cdots,\ e_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

Using these vectors, we easily know that Ae_1 is the first column of A and Ae_i is the ith column of A. So, in other words, we can say Col A =Span$\{Ae_1, Ae_2, \cdots, Ae_n\}$.

[1] Even though we need to specify where those $e_1, e_2, \cdots, e_n$ belong to, we just abuse the notations.

2.1 Elementary row operations

[2]We will skip the part that **every elementary row operation has a particular relation with an elementary matrix**, that is,

Doing an elementary row operation to A = Converting A into EA where E is the corresponding matrix.

As we have seen before, Ae_i represents the ith column of A. Hence, if we have a relation like

$$3Ae_1 + 4Ae_2 - Ae_3 + Ae_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

,then we also have

$$3EAe_1 + 4EAe_2 - EAe_3 + EAe_n = E(3Ae_1 + 4Ae_2 - Ae_3 + Ae_n) = E\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

From this example, we can figure out that **elementary row operations do not change linear dependence relation of column vectors**.

However, it might change Col A. In other words, there is **no reason** for

$$\text{Col } A = \text{Col } EA.$$

We also have some counterexamples :

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The column space of the former matrix contains $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ obviously, but the column space of the latter matrix does not contain that vector.

[2]As we have discussed this thing many times,

2.2 Nul A

We define Nul A as the set

$$\text{Nul } A = \{\mathbf{u} \in \mathbb{R}^n : A\mathbf{u} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}\}.$$

Conventionally, we use the notation $\mathbf{0}$ or $\vec{0}$ for the zero vector $\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$.

Is Nul A a subspace of $\mathbb{R}^n$?

The answer is yes. Because...

First of all, Nul A is obviously **a subset of** $\mathbb{R}^n$.

1) The zero vector : Since $A \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} = \mathbf{0}$, we can get $\begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \in$Nul A.

2) Closed under addition : Suppose that $\mathbf{u}$ and $\mathbf{v}$ is in Nul A. In other words, we are assuming that $A\mathbf{u} = \mathbf{0}$ and $A\mathbf{v} = \mathbf{0}$. We want to check whether or not $\mathbf{u}+\mathbf{v}$ is in Nul A. However, $A(\mathbf{u}+\mathbf{v}) = A\mathbf{u}+A\mathbf{v} = \mathbf{0}+\mathbf{0} = \mathbf{0}$. Hence, $\mathbf{u}+\mathbf{v} \in$ Nul A.

3) Closed under scalar multiplication :

As we have proven that 1), 2), and 3) are all true, Nul A is a subspace of $\mathbb{R}^n$.

Could Nul A and Col A be the same?

Yes, it sometimes happens. We have an example :

$$A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

has its Col A as Span$\left\{\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}$. At the same time, Nul A is Span$\left\{\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}$.

However, in general, Col $A \in \mathbb{R}^m$ and Nul $A \in \mathbb{R}^n$, so they couldn't be the same unless $m = n$ since they belong to different vector spaces unless $m = n$.

2.3 The Rank Theorem

Let A be an $m \times n$ matrix. Then,

$$\dim \text{Nul } A + \dim \text{Col } A = n.$$

In other words, we can change $\dim \text{Col } A$ into rank A.

Prove it!

3 Linear Transformation

As we have gone through several times, what it means by for a **map** or a **function** T being a linear transformation from a vector space V to a vector space W is

1) $T(v_1 + v_2) = T(v_1) + T(v_2)$ for all v_1, $v_2 \in V$.

2) $T(cv) = cT(v)$ for all $v \in V$ and $c \in \mathbb{R}$.

There is one fact that we can easily get from the definition : set $c = 0$ in 2), then we get $T(0) = 0$.

We have learned one good example of a linear transformation : Matrix multiplication. For example, let's think about a matrix

$$A = \begin{pmatrix} 1 & 5 & 2 \\ 2 & 0 & -3 \end{pmatrix}.$$

As we multiply a vector in $\mathbb{R}^3$, we get a vector in $\mathbb{R}^2$ like this ;

$$A \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 5 \end{pmatrix}, \quad A \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ 6 \end{pmatrix}, \quad A \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 8 \\ -1 \end{pmatrix}$$

So this matrix A gives a function from $\mathbb{R}^3$ to $\mathbb{R}^2$.[3] For this function to be a **linear transformation**, we require two conditions.

1) $A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v}$ for all $\mathbf{u}$, $\mathbf{v} \in \mathbb{R}^3$.

2) $Ac\mathbf{u} = cA\mathbf{u}$ for all $\mathbf{u} \in \mathbb{R}^3$ and $c \in \mathbb{R}$.

Those two conditions are trivially true in this case.

In this sense, every $m \times n$ matrix gives a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$. Here, let's recall Col A and Nul A.

$$\text{Col } A = \{A\mathbf{u} : \mathbf{u} \in \mathbb{R}^n\} \quad \& \quad \text{Nul } A = \{\mathbf{u} \in \mathbb{R}^n : A\mathbf{u} = \mathbf{0}\}$$

Similarly, for a linear transformation T from V to W, we define im T and $\ker T$ as following :

$$\text{im } T = \{T(v) : v \in V\} \quad \& \quad \ker T = \{v \in V : T(v) = 0\}$$

Then, im $T =$ Col A and $\ker T =$Nul A when T is the linear transformation defined by the matrix multiplication of A.

Remember that

Col A is a subspace of $\mathbb{R}^m$ and Nul A is a subspace of $\mathbb{R}^n$.

Similarly,

im T is a subspace of W and $\ker T$ is a subspace of V.

How could you figure out that fact?

[3]Note that $\mathbb{R}^3$ and $\mathbb{R}^2$ are vector spaces.

Let's take a look at another example.

Let T be the map from $\mathbb{P}_3$ to $\mathbb{R}$ defined as

$$T(\mathbf{p}(t)) = \mathbf{p}(1).$$

Note that $\mathbb{P}_3$ is a vector space and $\mathbb{R}$ is also a vector space. Hence, T is a map from a vector space to a vector space. In fact, $T : \mathbb{P}_3 \to \mathbb{R}$ is a linear transformation. In order to show that we need two conditions.

1) For every v_1 and $v_2 \in \mathbb{P}_3$, $T(v_1 + v_2) = T(v_1) + T(v_2)$?

2) For every $v \in \mathbb{P}_3$ and $c \in \mathbb{R}$, $T(cv) = cT(v)$?

Another way to show that

$$S = \{\mathbf{p}(t) \in \mathbb{P}_4 : \mathbf{p}(1) = 0\} \text{ is a subspace of } \mathbb{P}_4.$$