1 Coordinate Systems

1.1 Unique Representation Theorem

For every basis $\mathcal{B} = \{\mathbf{b}_1, \cdots, \mathbf{b}_n\}$ for a vector space V, a vector $v \in V$ has a unique represntation of the form

$$v = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \cdots + c_n\mathbf{b}_n$$

and we define

$$[v]_{\mathcal{B}} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$$

as the **coordinates of v relative to the basis $\mathcal{B}$.**

The coordinate mapping

$$v \mapsto [v]_{\mathcal{B}}$$

is a one-to-one linear transformation from V onto $\mathbb{R}^n$.

2 The dimension

As we have proved in the class, the dimension of a vector space V is well-defined. This implies many things.

3 The Rank Theorem

For every $m \times n$ matrix A,

$$\text{rank } A + \dim \text{Nul } A = n$$

1. Consider the polynomials $\mathbf{p}_1(t) = 1 + t^2$ and $\mathbf{p}_2(t) = 1 - t^2$. Is $\{\mathbf{p}_1, \mathbf{p}_2\}$ a linearly independent set in $\mathbb{P}_3$? Why or why not?

2. For each subspace below, find a basis for the subspace and state the dimension.

   a. $$\left\{ \begin{pmatrix} s - 2t \\ s + t \\ 3t \end{pmatrix} : s,\ t \in \mathbb{R} \right\}$$

   b. $$\left\{ \begin{pmatrix} p + 2q \\ -p \\ 3p - q \\ p + q \end{pmatrix} : p,\ q \in \mathbb{R} \right\}$$

   c. $$\left\{ \begin{pmatrix} 3a - c \\ -b - 3c \\ -7a + 6b + 5c \\ -3a + c \end{pmatrix} : s,\ t \in \mathbb{R} \right\}$$

3. Find the coordinate vector $[v]_{\mathcal{B}}$ of v relative to the given basis $\mathcal{B} = \{\mathbf{b}_1, \cdots, \mathbf{b}_n\}$.

   a. $$\mathbf{b}_1 = \begin{pmatrix} 1 \\ -1 \\ -3 \end{pmatrix}, \quad \mathbf{b}_2 = \begin{pmatrix} -3 \\ 4 \\ 9 \end{pmatrix}, \quad \mathbf{b}_3 = \begin{pmatrix} 2 \\ -2 \\ 4 \end{pmatrix}, \quad v = \begin{pmatrix} 8 \\ -9 \\ 6 \end{pmatrix}$$

   b. $$\mathbf{b}_1 = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}, \quad \mathbf{b}_2 = \begin{pmatrix} 2 \\ 0 \\ 8 \end{pmatrix}, \quad \mathbf{b}_3 = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}, \quad v = \begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix}$$

4. Let $\mathbf{p}_1(t) = 1 + t^2$, $\mathbf{p}_2(t) = t - 3t^2$, $\mathbf{p}_3(t) = 1 + t - 3t^2$. Use coordinate vectors to show that these polynomials form a basis for $\mathbb{P}_2$.

5. Mark each statement True or False. Justify the answer. (V is a vector space.)

   a. A plane in $\mathbb{R}^3$ is a two-dimensional subspace of $\mathbb{R}^3$.

   b. The dimension of the vector space $\mathbb{P}_4$ is 4.

   c. If $\dim V = n$ and S is a linearly independent set in V, then S is a basis for V.

   d. If a set $\{v_1, \cdots, v_p\}$ spans a finite-dimensional vector space V and if T is a set of more than p vectors in V, then T is linearly dependent.

   e. A vector space is infinite-dimensional if it is spanned by an infinite set.

   f. The only three-dimensional subspace of $\mathbb{R}^3$ is $\mathbb{R}^3$ itself.

6. Explain why the space of $\mathbb{P}$ of all polynomials is an infinite dimensional space.

7. Mark each statement True or False. Justify your answer. (V is a nonzero finite-dimensional vector space, and the vectors listed belong to V.)

   a. If there exists a set $\{v_1, \cdots, v_p\}$ that spans V, then $\dim V \leq p$.

   b. If there exists a linearly independent set $\{v_1, \cdots, v_p\}$ in V, then $\dim V \geq p$.

   c. If $\dim V = p$, then there exists a spanning set of $p + 1$ vectors in V.

   d. If every set of p elements in V fails to span V, then $\dim V > p$.