1 Vector Spaces and Linear Transformations (revisit)

1.1 More general linear transformations

A ***linear transformation*** T from a vector space V into a vector space W is a rule that assigns to each vector $\mathbf{x}$ in V a unique vector $T(\mathbf{x})$ in W, such that

(i) $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}$, $\mathbf{v}$ in V, and

(ii) $T(c\mathbf{u}) = cT(\mathbf{u})$ for all $\mathbf{u}$ in V and all scalars c.

Now, we define $\ker T$ as following :

$$\ker T = \{\mathbf{x} \in V | T(\mathbf{x}) = \mathbf{0}\}$$

So it is a generalized form of Nul A. Similarly, we define a concept similar to Col A. We call it as the **range of** T, denoted by Im T.[1]

1.2 Linearly independent sets & Bases

An indexed set of vectors $\{v_1, \cdots, v_n\}$ in V is said to be ***linearly independent*** if the vector equation

$$c_1 v_1 + \cdots + c_n v_n = 0$$

has only the trivial solution $c_1 = 0$, $\cdots$, $c_n = 0$. It is *linearly dependent* if it is not *linearly independent.*

Given H a subspace of a vector space V. An indexed set of vectors $\mathcal{B} = \{\mathbf{b}_1, \cdots, \mathbf{b}_n\}$ in V is a ***basis of*** H if (i) $\mathcal{B}$ is linearly independent, and (ii) $H = \text{Span}\{\mathbf{b}_1, \cdots, \mathbf{b}_n\}$.

1.2.1 A basis for $\mathbb{P}_n$

$$S = \{1, t, \cdots, t^n\} \text{ is a basis of } \mathbb{P}_n.$$

1.3 Theorem 5

Let $S = \{v_1, \cdots, v_n\}$ be a set in V, and let $H = \text{Span}\{v_1, \cdots, v_n\}$.

a. If one of the vectors in S, say v_k, is a linear combination of the remaining vectors in S, then the set formed from S by removing v_k still spans H.

b. If $H \neq \{\mathbf{0}\}$, some subset of S is a basis for H.

1.4 Theorem 6

The pivot columns of a matrix A form a basis for Col A.

[1] In the textbook, the author does not use ker or Im at all. However, for the sake of convenience, in this class we will use that notations.

1. Either use an appropriate theorem to show that the given set, W, is a vector space, or find a specific example to the contrary.

   a. $\left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} : a + b + c = 2 \right\}$

   b. $\left\{ \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} : \begin{matrix} 3a + b = c \\ a + b + 2c = 2d \end{matrix} \right\}$

   c. $\left\{ \begin{pmatrix} 3p - 5q \\ 4q \\ p \\ q + 1 \end{pmatrix} : p,\ q \text{ real} \right\}$

   d. $\left\{ \begin{pmatrix} -s + 3t \\ s - 2t \\ 5s - t \end{pmatrix} : s,\ t \text{ real} \right\}$

2. Mark each statement True or False. Justify each answer.

   a. The null space of A is the solution set of the equation $A\mathbf{x} = \mathbf{0}$.

   b. The null space of an $m \times n$ matrix is in $\mathbb{R}^m$.

   c. $\ker T$ is a vector space when T is a linear transformation.

   d. Col A is the set of all solutions of $A\mathbf{x} = \mathbf{b}$.

   e. Im T is a vector space when T is a linear transformation.

3. Mark each statement True or False. Justify each answer.

   a. If $\mathbf{f}$ is a function in the vector space V of all real-valued functions on $\mathbb{R}$ and if $\mathbf{f}(t) = 0$ for some t, then $\mathbf{f}$ is the zero vector in V.

   b. A subspace is also a vector space.

   c. A vector is any element of a vector space.

   d. A vector space is also a subspace.

   e. $\mathbb{R}^2$ is a subspace of $\mathbb{R}^3$.

4. Find a basis for the set of vectors in $\mathbb{R}^3$ in the plane $x - 3y + 2z = 0$.

5. Assume that A is row equivalent to B. Find bases for Nul A and Col A.

   a.
$$A = \begin{pmatrix} -2 & 4 & -2 & -4 \\ 2 & -6 & -3 & 1 \\ -3 & 8 & 2 & -3 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 6 & 5 \\ 0 & 2 & 5 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

   b.
$$A = \begin{pmatrix} 1 & 2 & 3 & -4 & 8 \\ 1 & 2 & 0 & 2 & 8 \\ 2 & 4 & -3 & 10 & 9 \\ 3 & 6 & 0 & 6 & 9 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 & 0 & 2 & 5 \\ 0 & 0 & 3 & -6 & 3 \\ 0 & 0 & 0 & 0 & -7 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

6. Let $\mathbf{v}_1 = \begin{pmatrix} 3 \\ 4 \\ -2 \\ -5 \end{pmatrix}$, $\mathbf{v}_1 = \begin{pmatrix} 4 \\ 3 \\ 2 \\ 4 \end{pmatrix}$, and $\mathbf{v}_1 = \begin{pmatrix} 2 \\ 5 \\ -6 \\ -14 \end{pmatrix}$. It can be verified that $2\mathbf{v}_1 - \mathbf{v}_2 - \mathbf{v}_3 = \mathbf{0}$. Use this information to find a basis for $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

7. Mark each statement True or False. Justify each answer.

   a. A single vector by itself is linearly dependent.

   b. The columns of an invertible $n \times n$ matrix form a basis for $\mathbb{R}^n$.

   c. A basis is a linearly independent set that is as large as possible.

   d. If B is an echelon form of a matrix A, then the pivot columns of B form a basis for Col A.

   e. If a finite set S of nonzero vectors spans a vector space V, then some subset of S is a basis for V.

8. Consider the polynomials $\mathbf{p}_1(t) = 1 + t^2$ and $\mathbf{p}_2(t) = 1 - t^2$. Is $\{\mathbf{p}_1, \mathbf{p}_2\}$ a linearly independent set in $\mathbb{P}_3$? Why or why not?