1 Properties of Determinants (Cont'd)

1.1 A linearity property of the determinant function

Remember that we have defined the *determinant of a matrix* as a function from a matrix to a real number. However, a matrix can be regarded as a collection of column vectors. So, $\det A = \det(A_1\ A_2\ \cdots\ A_n)$. In this viewpoint, det is **linear** for each columns. It means

$$\det(A_1\ A_2\ \cdots\ (u+v)\ \cdots\ A_n) = \det(A_1\ A_2\ \cdots\ u\ \cdots\ A_n) + \det(A_1\ A_2\ \cdots\ v\ \cdots\ A_n)$$

and

$$\det(A_1\ A_2\ \cdots\ cu\ \cdots\ A_n) = c\det(A_1\ A_2\ \cdots\ u\ \cdots\ A_n)$$

2 Cramer's Rule, Volume, and Linear Transformations

2.1 Cramer's Rule ; Find the solution $\mathbf{x}$ satisfying $A\mathbf{x} = \mathbf{b}$

Remember that when A is invertible, there exists a unique solution $\mathbf{x}$ satisfying $A\mathbf{x} = \mathbf{b}$. Cramer's Rule lets you find each entries of $\mathbf{x}$, that is, x_i's explicitly. It is given as :

$$x_i = \frac{\det A_i(\mathbf{b})}{\det A}$$

Here, $A_i(\mathbf{b})$ is the matrix obtained from A by **replacing the ith column by the vector $\mathbf{b}$**.

2.2 An application of Cramer's Rule to find the inverse matrix A^{-1}

If $B = (\mathbf{x}_1\ \mathbf{x}_2\ \cdots\ \mathbf{x}_n)$ is the inverse matrix ($\mathbf{x}_i$'s are vectors) then $AB = I_n$. This means that $A\mathbf{x}_j = \mathbf{e}_j$. (Note that $\mathbf{e}_j$ is the vector $(0\ 0\ \cdots\ 0\ 1\ 0\ \cdots\ 0)$ where 1 is located at the jth entry.) Hence, A^{-1} has its jth column as $\frac{\det A_i(\mathbf{e}_j)}{\det A}$.

2.3 det=Volume (revisit)

Theorem 10 says that given a parallelogram (or parallelepiped) S,

$$\{\text{area (or volume) of } T(S)\} = |\det A| \{\text{area (or volume) of } S\}$$

where A is the matrix corresponding to the linear transformation T.

3 Vector Spaces

A vector space is a nonempty set V of vectors with two operations (addition and scalar multiplication) subject to the ten axioms. What the list those 10 axioms is exactly is not much important. However, they basically contain the properties of $\mathbb{R}^n$. *Please note that* ***mathematicians*** *love to* ***generalize*** *or* ***abstractify*** *some examples to generalized ones.*

3.1 Examples of Vector Spaces

3.1.1 the set of polynomials of degree at most n, denoted by $\mathbb{P}_n$

3.1.2 the set of all real-valued functions defined on a set D

1. Use Cramer's rule to compute the solutions of the systems.

   a. $$\begin{aligned} 5x_1 + 7x_2 &= 3 \\ 2x_1 + 4x_2 &= 1 \end{aligned}$$

   b. $$\begin{aligned} -5x_1 + 3x_2 &= 9 \\ 3x_1 - x_2 &= -5 \end{aligned}$$

   c. $$\begin{aligned} 2x_1 + x_2 + x_3 &= 4 \\ -x_1 + 2x_3 &= 2 \\ 3x_1 + x_2 + 3x_3 & \end{aligned}$$

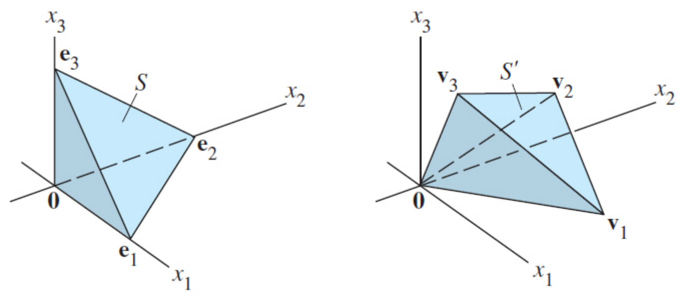
2. Find the volume of the parallelepiped with one vertex at the origin and adjacent vertices at $(1, 4, 0)$, $(-2, -5, 2)$, and $(-1, 2, -1)$.

3. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation determined by the matrix $A = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$, where a, b, and c are positive numbers. Let S be the unit ball, whose bounding surface has the equation $x_1^2 + x_2^2 + x_3^2 = 1$.

   a. Show that $T(S)$ is bounded by the ellipsoid with the equation $\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} + \frac{x_3^2}{c^2} = 1$.

   b. Use the fact that the volume of the unit ball is $4\pi/3$ to determine the volume of the region bounded by the ellipsoid in part (a).

4. Let S be the tetrahedron in $\mathbb{R}^3$ with vertices at the vectors $\mathbf{0}$, $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$, and let S' be the tetrahedron with vertices at vectors $\mathbf{0}$, $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$. See the figure.



a. Describe a linear transformation that maps S onto S'.

b. Find a formula for the volume of the tetrahedron S' using the fact that

$$\{\text{volume of } S\} = (1/3)\{\text{area of base}\} \cdot \{\text{height}\}$$

5. Mark each statement True or False. Justify each answer. Assume that all matrices here are square.

a. If A is a 2×2 matrix with a zero determinant, then one column of A is a multiple of the other.

b. If A is a 3×3 matrix, then $\det 5A = 25 \det A$.

6. Determine if the given set is a subspace of $\mathbb{P}_n$ for an appropriate value of n. Justify your answers.

a. All polynomials of the form $\mathbf{p}(t) = at^2$, where a is in $\mathbb{R}$.

b. All polynomials of degree at most 3, with integers as coefficients.

c. All polynomials in $\mathbb{P}_n$ such that $\mathbf{p}(0) = 0$.