Let's first start from *Matrix Operations*, especially *matrix multiplication*. Given an $n \times m$ matrix $A = (a_{ij})$ and an $m \times l$ matrix $B = (b_{ij})$, we define the product of two matrices A and B, denoted by AB, as the matrix whose (i, j)-entry is a kind of product of ith row of A and jth column of B.

This gives the same matrix AB as the matrix we've defined previously. There is one important and seemingly obvious property

$$(AB)C = A(BC)$$

If we write down m real numbers vertically (it is an m-dimensional vector) then we have a $m \times 1$ matrix. So, we can regard an m-dimensional vector as an $m \times 1$ matrix. In this viewpoint, as a particular case of this multiplication, we have a product of a matrix and a vector.

For example, let $A = \begin{pmatrix} 3 & 2 & -1 & 7 \\ 2 & -5 & 0 & 1 \\ 4 & 1 & -2 & 3 \end{pmatrix}$ and $\mathbf{x}$ be a 4-dimensional vector whose entries are 5, 0, -1, 1, that is $\mathbf{x} = \begin{pmatrix} 5 \\ 0 \\ -1 \\ 1 \end{pmatrix}$. Then, their product results in

$$\begin{pmatrix} 3 & 2 & -1 & 7 \\ 2 & -5 & 0 & 1 \\ 4 & 1 & -2 & 3 \end{pmatrix} \begin{pmatrix} 5 \\ 0 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 23 \\ 11 \\ 25 \end{pmatrix}$$

since

$$\begin{aligned} 3 \cdot 5 + 2 \cdot 0 + (-1) \cdot (-1) + 7 \cdot 1 &= 23 \\ 2 \cdot 5 + (-5) \cdot 0 + 0 \cdot (-1) + 1 \cdot 1 &= 11 \\ 4 \cdot 5 + 1 \cdot 0 + (-2) \cdot (-1) + 3 \cdot 1 &= 25 \end{aligned}$$

Now, let's make a linear system as below :

$$\begin{aligned} 3x_1 + 2x_2 - x_3 + 7x_4 &= 23 \\ 2x_1 - 5x_2 + x_4 &= 11 \\ 4x_1 + x_2 - 2x_3 + 3x_4 &= 25 \end{aligned}$$

In a form of a matrix equation, it is

$$\begin{pmatrix} 3 & 2 & -1 & 7 \\ 2 & -5 & 0 & 1 \\ 4 & 1 & -2 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 23 \\ 11 \\ 25 \end{pmatrix}$$

This equation has a form of $A\mathbf{x} = \mathbf{b}$. This is the connection between a matrix multiplication and a linear system.

As I have mentioned in the first class, one of our main goals in this class is to determine which kind of linear system is consistent or inconsistent and to figure out how many solutions exist for a given linear system and furthermore how we can write down all solutions explicitly.

$$\begin{pmatrix} A_1 & A_2 & \cdots & A_n \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \mathbf{b} \end{pmatrix}$$

In a vector equation form, we can convert the above equation into

$$x_1 A_1 + x_2 A_2 + \cdots + x_n A_n = \mathbf{b}$$

Here is the point where we define a linear combination of vectors $\{A_1, A_2, \cdots, A_n\}$. So, we can state that

$$A\mathbf{x} = \mathbf{b} \text{ has a solution } \mathbf{x} \text{ if } \mathbf{b} \text{ is a linear combination of } \{A_1, A_2, \cdots, A_n\}$$

Keep this in your mind! We will come back here in a minute.

One way to solve for $\mathbf{x}$ is using row reductions (elementary row operations). If you take a careful look at how elementary row operations are implemented, you can easily find that those row operations correspond to some matrices. Let's first think about a linear system (= a matrix equation)

$$\begin{pmatrix} 1 & 0 & 2 & -2 \\ 3 & 2 & 4 & 8 \\ 0 & 3 & -7 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}$$

Let's multiply a matrix $\begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ left of each sides. Then, we get a new matrix equation

$$\begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 & -2 \\ 3 & 2 & 4 & 8 \\ 0 & 3 & -7 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}$$

Since $(AB)C = A(BC)$, we can compute the left hand side calculating the multiplication of left two matrices. That gives

$$\begin{pmatrix} 1 & 0 & 2 & -2 \\ 0 & 2 & -2 & 14 \\ 0 & 3 & -7 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix}$$

Here, note that the second row has been changed into *the second row*$-3\times$*the first row*. From this calculation we can find that the matrix that all digonal entries equal to 1 and (i, j)-entry is a for some i and j and all the other entries are equal to zero corresponds to the elementary operation that changes **the jth row** into **the jth row$-a\times$the ith row**. Let's think about the next step to solve the given matrix equation.

Multiplying $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$ left of each sides gives

$$\begin{pmatrix} 1 & 0 & 2 & -2 \\ 0 & 1 & -1 & 7 \\ 0 & 3 & -7 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix}$$

As a next step, what should we do? we need to change the third row into itself$-3\times$the second row. What is the matrix corresponding to this elementary row operation? It is $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{pmatrix}$

$$\begin{pmatrix} 1 & 0 & 2 & -2 \\ 0 & 1 & -1 & 7 \\ 0 & 0 & -4 & -16 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix}$$

Now, you know what the final step is! The most reduced form of this equation will be

$$\begin{pmatrix} 1 & 0 & 0 & -10 \\ 0 & 1 & 0 & 11 \\ 0 & 0 & 1 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix}$$

Therefore, we can explicitly write down all solutions

$$\begin{aligned} x_1 &= 10x + 3 \\ x_2 &= -11x \\ x_3 &= -4x - 1 \\ x_4 &= x \end{aligned}$$

Or

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 10 \\ -11 \\ -4 \\ 1 \end{pmatrix} x + \begin{pmatrix} 3 \\ 0 \\ -1 \\ 0 \end{pmatrix}$$

You'd be better to remember that the first vector $\begin{pmatrix} 10 \\ -11 \\ 4 \\ 1 \end{pmatrix}$ is a solution for $A\mathbf{x} = \mathbf{0}$ and $\begin{pmatrix} 3 \\ 0 \\ -1 \\ 0 \end{pmatrix}$ is a particular solution for $A\mathbf{x} = \mathbf{b}$.

A's reduced echelon form has a pivot position in every row and this assures that there exist solutions for the given matrix equation $A\mathbf{x} = \mathbf{b}$ whatever $\mathbf{b}$ is. If the last row of the reduced echelon form was all zeros and the last entry of $E\mathbf{b}$ was nonzero then this system could be inconsistent. Hence, if there is a row that has no pivot position then the linear transformation is not onto.

Now, let's move on to **onto**. Recall that a matrix transformation $\mathbf{x} \mapsto A\mathbf{x}$ is called onto if $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b} \in \mathbb{R}^m$. As we have seen right before, $\mathbf{x} \mapsto A\mathbf{x}$ is onto if and only if there is a pivot position in every row. These statements are also equivalent to **onto**ness of the transformation.

a. For each $\mathbf{b}$ in $\mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution.

b. Each $\mathbf{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of A.

c. The columns of A span $\mathbb{R}^m$. (Col $A = \mathbb{R}^m$)

d. The rank of A is m.

Similarly, $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one if and only if one of these followings is satisfied :

a. $A\mathbf{x} = \mathbf{0}$ has a unique solution $\mathbf{x} = \mathbf{0}$.

b. Nul $A = 0$

c. rank $A = n$

Especially, when A is a square matrix we can say that Col $A = \mathbb{R}^n$. This results in that *the columns of A form a basis of* $\mathbb{R}^n$.