1 Subspaces of \mathbb{R}^n

1.1 Which set is called called a subspace of \mathbb{R}^n ?

Given a set of vectors $\{v_1, \dots, v_n\}$, we have defined $\text{Span}\{v_1, \dots, v_n\}$. This gives an exaple of a subspace!

1.2 Basis for a subspace

For a given subspace H of \mathbb{R}^n , a **basis** for H is a linearly independent set in H that spans H, that is, Span(the set) = H.

1.3 Dimension of a subspace

For a given nonzero subspace H, we denote dimH as the *dimension* of the number of vectors in any basis for H. In order to define the concept of the *dimension* we need to prove that the number of vectors in two different bases for H are always same. How?

2 Column space and Null space of a Matrix

Column Space of a matrix A (Col A):

Null Space of a matrix A (Nul A):

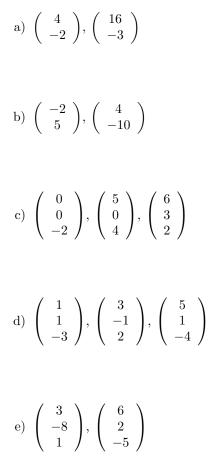
Given a $m \times n$ matrix A, the column space of A and the null space of A are both examples of subspaces. Why?

2.1 Rank of a matrix

The *rank* of a matrix A, denoted by rank A, is defined as the *dimension* of the column space of A. We have one important theorem named **The Rank Theorem**;

rank $A + \dim \operatorname{Nul} A = n$, where A is an $n \times n$ matrix

Exercise 1. Determine which sets below are bases for \mathbb{R}^2 or \mathbb{R}^3 . Justify your answer.



Exercise 2. If B is a 7×7 matrix and Col $B = \mathbb{R}^7$, what can be said about solutions of equations of the form $B\mathbf{x} = \mathbf{b}$ for $\mathbf{b} \in \mathbb{R}^7$?

Exercise 3. Suppose a 4×6 matrix A has four pivot columns. Is Col $A = \mathbb{R}^4$? Is Nul $A = \mathbb{R}^2$? Explain your answers.

Exercise 4. Suppose a 4×7 matrix A has three pivot columns. Is Col $A = \mathbb{R}^3$? What is the dimension of Nul A? Explain your answers.

Exercise 5. Justify each answer.

a) There exists a 3×5 matrix A such that dim Nul A = 3 and dim Col A = 2.

b) Show that a set $\{v_1, \dots, v_5\}$ in \mathbb{R}^n is linearly independent if dim $\text{Span}\{v_1, \dots, v_5\} = 4$.

c) Let A be an $n \times p$ matrix whose column space is p-dimensional. Explain why the columns of A must be linearly independent.