Change of Basis

Consider two bases for vector space V:

$$\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_n\}, \quad \mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2, \cdots, \mathbf{c}_n\}.$$

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change-of-coordinates matrix: $\underset{\mathcal{C} \leftarrow \mathcal{B}}{\mathbf{P}} \stackrel{def}{=} [[\mathbf{b}_1]_{\mathcal{C}}, [\mathbf{b}_2]_{\mathcal{C}}, \cdots, [\mathbf{b}_n]_{\mathcal{C}}].$

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Thm: For any vector $\mathbf{x} \in V$, $[\mathbf{x}]_{\mathcal{C}} = \underset{\mathcal{C} \leftarrow \mathcal{B}}{\mathbf{P}} \; [\mathbf{x}]_{\mathcal{B}}$

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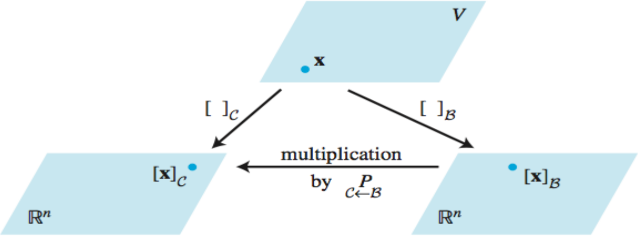

FIGURE 2 Two coordinate systems for V.

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Consider two bases for vector space V:

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PROOF: Let $\mathbf{x} = \alpha_1 \mathbf{b}_1 + \cdots + \alpha_n \mathbf{b}_n$ so that $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}.$

Change of Basis

Consider two bases for vector space V:

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Change of Basis

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$$[\mathbf{x}]_{\mathcal{C}} = [\alpha_1 \mathbf{b}_1 + \cdots + \alpha_n \mathbf{b}_n]_{\mathcal{C}} = \alpha_1 [\mathbf{b}_1]_{\mathcal{C}} + \cdots + \alpha_n [\mathbf{b}_n]_{\mathcal{C}}$$

Change of Basis

Consider two <u>bases</u> for vector space V:

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change-of-coordinates matrix: $\underset{\mathcal{C} \leftarrow \mathcal{B}}{\mathbf{P}} \stackrel{def}{=} \left[[\mathbf{b}_1]_{\mathcal{C}}, [\mathbf{b}_2]_{\mathcal{C}}, \cdots, [\mathbf{b}_n]_{\mathcal{C}} \right].$

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Consider two bases for vector space V:

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EX: Consider two bases for $\mathcal{R}^2$: $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}\,, \quad \mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}\,.$

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with $\quad \mathbf{b}_1 = \begin{bmatrix} -9 \\ 1 \end{bmatrix}, \ \mathbf{b}_2 = \begin{bmatrix} -5 \\ -1 \end{bmatrix}, \ \mathbf{c}_1 = \begin{bmatrix} 1 \\ -4 \end{bmatrix}, \ \mathbf{c}_2 = \begin{bmatrix} 3 \\ -5 \end{bmatrix}.$

Find $\quad \underset{\mathcal{C} \leftarrow \mathcal{B}}{\mathbf{P}} = \left[[\mathbf{b}_1]_{\mathcal{C}}, [\mathbf{b}_2]_{\mathcal{C}} \right]$

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Find $\underset{\mathcal{C} \leftarrow \mathcal{B}}{\mathbf{P}} = [[\mathbf{b}_1]_{\mathcal{C}}, [\mathbf{b}_2]_{\mathcal{C}}]$

SOLUTION: Let $[\mathbf{b}_1]_{\mathcal{C}} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$. Then

$\mathbf{b}_1 = \alpha_1 \mathbf{c}_1 + \alpha_2 \mathbf{c}_2$

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Find $\quad \mathcal{C} \overset{\mathbf{P}}{\leftarrow} \mathcal{B} \; = [[\mathbf{b}_1]_{\mathcal{C}}\, , [\mathbf{b}_2]_{\mathcal{C}}]$

SOLUTION: Let $[\mathbf{b}_1]_{\mathcal{C}} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$. Then

$$\mathbf{b}_1 = \alpha_1 \, \mathbf{c}_1 + \alpha_2 \, \mathbf{c}_2 = \begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$$

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Find $\underset{\mathcal{C} \leftarrow \mathcal{B}}{\mathbf{P}} = [[\mathbf{b}_1]_\mathcal{C}, [\mathbf{b}_2]_\mathcal{C}]$

SOLUTION: Let $[\mathbf{b}_1]_\mathcal{C} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$. Then

$$\mathbf{b}_1 = \alpha_1 \mathbf{c}_1 + \alpha_2 \mathbf{c}_2 = \begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \Longrightarrow \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 \end{bmatrix}^{-1} \mathbf{b}_1.$$

therefore $[\mathbf{b}_1]_\mathcal{C} = \begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 \end{bmatrix}^{-1} \mathbf{b}_1$, and $[\mathbf{b}_2]_\mathcal{C} = \begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 \end{bmatrix}^{-1} \mathbf{b}_2$.

$$\begin{aligned} \underset{\mathcal{C} \leftarrow \mathcal{B}}{\mathbf{P}} &= [[\mathbf{b}_1]_\mathcal{C}, [\mathbf{b}_2]_\mathcal{C}] = \begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 3 \\ -4 & -5 \end{bmatrix}^{-1} \begin{bmatrix} -9 & -5 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 6 & 4 \\ -5 & -3 \end{bmatrix} \end{aligned}$$

§5.1 Eigenvalues and Eigenvectors

DEF: Given $A \in \mathcal{R}^{n \times n}$, the **eigenvalue** and **eigenvector** are a pair of scalar λ and non-zero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$.

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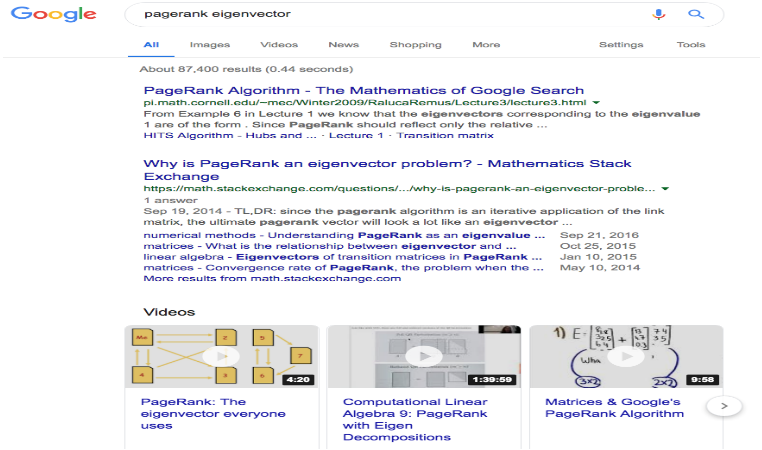
Example: $A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\lambda = 2$.

$$A\mathbf{x} = \lambda\mathbf{x} : \quad \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

§5.1 Eigenvalues and Eigenvectors



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pi.math.cornell.edu/~mec/Winter2009/RalucaRemus/Lecture3/lecture3.html
From Example 6 in Lecture 1 we know that the **eigenvectors** corresponding to the **eigenvalue** 1 are of the form . Since **PageRank** should reflect only the relative ...
HITS Algorithm - Hubs and ... · Lecture 1 · Transition matrix

Why is PageRank an eigenvector problem? - Mathematics Stack Exchange
https://math.stackexchange.com/questions/.../why-is-pagerank-an-eigenvector-proble...
1 answer
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matrices - What is the relationship between **eigenvector** and ...	Oct 25, 2015
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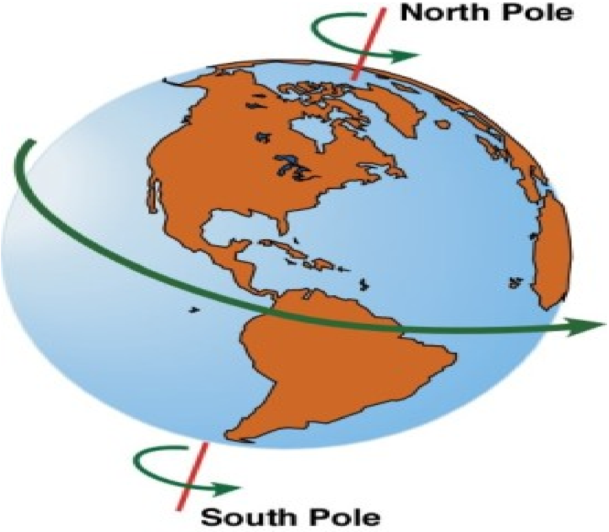
§5.1 Eigenvalues and Eigenvectors



Example: Let $A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$. An eigenvalue of A is $\lambda = 2$.

Find the corresponding eigenvector.

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$$\left(\begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \right) \mathbf{x} = \mathbf{0}, \quad \Longrightarrow \begin{bmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{bmatrix} \mathbf{x} = \mathbf{0}.$$

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$$\mathbf{x} = x_1 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 6 \\ 1 \end{bmatrix}.$$

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$$\left(\begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \right) \mathbf{x} = \mathbf{0}, \implies \begin{bmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{bmatrix} \mathbf{x} = \mathbf{0}.$$

$$\mathbf{x} = x_1 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 6 \\ 1 \end{bmatrix}.$$

Eigenvector is <u>any non-zero</u> vector in **Span** $\left(\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 6 \\ 1 \end{bmatrix} \right)$

Eigenspace

Given $A \in \mathcal{R}^{n \times n}$, the **eigenvalue**, **eigenvector** pair satisfies

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$$\Uparrow$$

DEF: **eigenspace** of A corresponding to λ

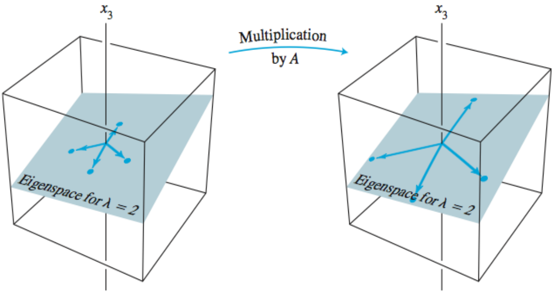

FIGURE 3 A acts as a dilation on the eigenspace.

Thm: If $A \in \mathbb{R}^{n \times n}$ is a matrix and $\lambda_1, \lambda_2, \cdots, \lambda_k$ are distinct eigenvalues of A with associated eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$, then $\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k\}$ is a linearly independent set.

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Proof: Assume Thm true for $k = s \geq 1$. For $k = s + 1 \geq 2$, let

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Then $\quad \alpha_1 \lambda_k \mathbf{v}_1 + \alpha_2 \lambda_k \mathbf{v}_2 + \cdots + \alpha_k \lambda_k \mathbf{v}_k = \mathbf{0} \quad (\ell_1)$

and

$$A(\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \cdots + \alpha_k \mathbf{v}_k)$$
$$= \alpha_1 \lambda_1 \mathbf{v}_1 + \alpha_2 \lambda_2 \mathbf{v}_2 + \cdots + \alpha_k \lambda_k \mathbf{v}_k = \mathbf{0}. \, (\ell_2)$$

Thm: If $A \in \mathbb{R}^{n\times n}$ is a matrix and $\lambda_1, \lambda_2, \cdots, \lambda_k$ are distinct eigenvalues of A with associated eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$, then $\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k\}$ is a linearly independent set.

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Then $$\alpha_1 \lambda_k \mathbf{v}_1 + \alpha_2 \lambda_k \mathbf{v}_2 + \cdots + \alpha_k \lambda_k \mathbf{v}_k = \mathbf{0} \quad (\ell_1)$$

and $$A(\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \cdots + \alpha_k \mathbf{v}_k)$$
$$= \alpha_1 \lambda_1 \mathbf{v}_1 + \alpha_2 \lambda_2 \mathbf{v}_2 + \cdots + \alpha_k \lambda_k \mathbf{v}_k = \mathbf{0}. \, (\ell_2)$$

Taking difference between (ℓ_1) and (ℓ_2),

$$\alpha_1 (\lambda_1 - \lambda_k) \mathbf{v}_1 + \alpha_2 (\lambda_2 - \lambda_k) \mathbf{v}_2 + \cdots + \alpha_{k-1} (\lambda_{k-1} - \lambda_k) \mathbf{v}_{k-1} = \mathbf{0}.$$

By induction, $\alpha_1 = \alpha_2 = \cdots = \alpha_{k-1} = 0$, and so $\alpha_k = 0$. □

- Let $A = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & 2 \\ 1 & -1 & 4 \end{pmatrix}$. Eigenvalues $3, 2, 2$, eigenvectors

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§5.2 Characteristic Equation

PROBLEM: Let $A = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$. Find eigenvalues of A

SOLUTION: Let λ be an eigenvalue of A:

$$(A - \lambda I)\,\mathbf{x} = \mathbf{0}, \quad \mathbf{x} \neq \mathbf{0}$$

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Matrix $\begin{bmatrix} 2-\lambda & 3 \\ 3 & -6-\lambda \end{bmatrix}$ must not be invertible.

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So eigenvalues are $\lambda_1 = 3, \lambda_2 = -7$.

$$(A - \lambda_1 I)\,\mathbf{x}_1 = \mathbf{0}, \quad \mathbf{x}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \quad (A - \lambda_2 I)\,\mathbf{x}_2 = \mathbf{0}, \quad \mathbf{x}_2 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

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PROBLEM: Let $A \in \mathcal{R}^{n \times n}$. Let λ be an eigenvalue of A:

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EX: Let $A = \begin{pmatrix} 5 & -2 & 6 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -1 & 4 \\ 0 & 0 & 0 & 3 \end{pmatrix}$.

Find the characteristic equation of A.

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Find the characteristic equation of A.

SOLUTION:

$$\begin{aligned} \mathbf{det}\,(A - \lambda I) &= \mathbf{det} \begin{pmatrix} 5-\lambda & -2 & 6 & 2 \\ 0 & 1-\lambda & 1 & 2 \\ 0 & 0 & -1-\lambda & 4 \\ 0 & 0 & 0 & 3-\lambda \end{pmatrix} \\ &= (5-\lambda)\,(1-\lambda)\,(-1-\lambda)\,(3-\lambda) \end{aligned}$$

Similarity

DEF: Let $A, B \in \mathcal{R}^{n \times n}$. A is **similar** to B if there exists an invertible matrix $P \in \mathcal{R}^{n \times n}$ such that

$$A = P\,B\,P^{-1}.$$

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Example: $A = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}, B = \begin{bmatrix} 3 & 0 \\ 0 & -7 \end{bmatrix}, P = \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix},$

then

$$P^{-1} = \frac{1}{10}\begin{bmatrix} 3 & 1 \\ -1 & 3 \end{bmatrix}, \quad A = P\,B\,P^{-1}.$$

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Thm: A and B have the same eigenvalues.
PROOF: Since $A - \lambda I = P\,B\,P^{-1} - \lambda I = P\,(B - \lambda I)\,P^{-1}$. We have

$$\begin{aligned} \mathbf{det}\,(A - \lambda I) &= \mathbf{det}\left(P\,(B - \lambda I)\,P^{-1}\right) \\ &= \mathbf{det}\,(P)\,\mathbf{det}\,(B - \lambda I)\,\mathbf{det}\left(P^{-1}\right) \\ &= \mathbf{det}\,(B - \lambda I) \end{aligned}$$

Therefore

$$\mathbf{det}\,(A - \lambda I) = 0 \iff \mathbf{det}\,(B - \lambda I) = 0.$$

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Both A and B share the same eigenvalues. Since B has eigenvalues $3, -7$, so does A.

AlphaZero: Board Game AI Superhuman Genius

- AlphaZero beats every human/AI in Chess/Go

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- AlphaZero built on Deep Reinforcement Learning / Markov Decision Process.

Markov Decision Process (MDP)

- set of states S, set of actions A, initial state S_0
- transition model P(s,a,s')
  - P([1,1], up, [1,2]) = 0.8
- reward function r(s)
  - r([4,3]) = +1
- goal: maximize cumulative reward in the long run

- policy: mapping from S to A
  - π(s) or π(s,a) (deterministic vs. stochastic)
- reinforcement learning
  - transitions and rewards usually not available
  - how to change the policy based on experience
  - how to explore the environment

EX: Let $A = \begin{bmatrix} 0.95 & 0.03 \\ 0.05 & 0.97 \end{bmatrix}$ and $\mathbf{x}_0 = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix}$, analyze the long-term behavior of the Markov Process

$$\mathbf{x}_{k+1} = A\mathbf{x}_k, \quad k = 0, 1, 2, \cdots, .$$

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SOLUTION:

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Therefore eigenvalues are $\lambda_1 = 1, \lambda_2 = 0.92$.

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$$(A - \lambda_1\, I)\,\mathbf{v}_1 = \mathbf{0}, \quad \mathbf{v}_1 = \begin{bmatrix} 3 \\ 5 \end{bmatrix}, \quad (A - \lambda_2\, I)\,\mathbf{v}_2 = \mathbf{0}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

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- Find coordinates for $\mathbf{x}_0$ in $\{\mathbf{v}_1, \mathbf{v}_2\}$ basis:

$$\mathbf{x}_0 = c_1\,\mathbf{v}_1 + c_2\,\mathbf{v}_2, \quad \Longrightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix}^{-1} \mathbf{x}_0 = \frac{1}{40}\begin{bmatrix} 5 \\ 9 \end{bmatrix}.$$

EX: Analyze the long-term behavior of the MARKOV PROCESS

$$\mathbf{x}_{k+1} = A\,\mathbf{x}_k, \quad k = 0, 1, 2, \cdots, .$$

SOLUTION: Eigenvalues are $\lambda_1 = 1, \lambda_2 = 0.92$.

$$A\,\mathbf{v}_1 = \lambda_1\,\mathbf{v}_1, \quad A\,\mathbf{v}_2 = \lambda_2\,\mathbf{v}_2, \quad \mathbf{v}_1 = \begin{bmatrix} 3 \\ 5 \end{bmatrix}.$$

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$$\begin{aligned}
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&\vdots \\
\mathbf{x}_k &= A\,\mathbf{x}_{k-1} = c_1\,\lambda_1^k\,\mathbf{v}_1 + c_2\,\lambda_2^k\,\mathbf{v}_2.
\end{aligned}$$

As $k \to \infty$, $\lambda_1^k = 1$, $\lambda_2^k \to 0$, and $\mathbf{x}_k \to c_1\,\mathbf{v}_1 = \frac{1}{8}\begin{bmatrix} 3 \\ 5 \end{bmatrix} \overset{def}{=} \mathbf{x}_{\text{stationary}}$.

EX: Matrix powers of a diagonal matrix. Let $D = \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}$. Then

$$\begin{aligned} D^2 &= D \cdot D = \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix} \cdot \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 3^2 & 0 \\ 0 & 5^2 \end{bmatrix}, \\ D^3 &= D \cdot D^2 = \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix} \cdot \begin{bmatrix} 3^2 & 0 \\ 0 & 5^2 \end{bmatrix} = \begin{bmatrix} 3^3 & 0 \\ 0 & 5^3 \end{bmatrix}, \\ &\vdots \\ D^k &= D \cdot D^{k-1} = \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix} \cdot \begin{bmatrix} 3^{k-1} & 0 \\ 0 & 5^{k-1} \end{bmatrix} = \begin{bmatrix} 3^k & 0 \\ 0 & 5^k \end{bmatrix}. \end{aligned}$$

EX: Find matrix powers of $A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$, given $A = P\,D\,P^{-1}$, with $P = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$, and $D = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$.

SOLUTION:

$$\begin{aligned} A^2 &= \left(P\,D\,P^{-1}\right)\cdot\left(P\,D\,P^{-1}\right) = P\,D^2\,P^{-1}, \\ A^3 &= \left(P\,D\,P^{-1}\right)\cdot\left(P\,D^2\,P^{-1}\right) = P\,D^3\,P^{-1}, \\ &\vdots \\ A^k &= \left(P\,D\,P^{-1}\right)\cdot\left(P\,D^{k-1}\,P^{-1}\right) = P\,D^k\,P^{-1} \\ &= \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} 5^k & 0 \\ 0 & 3^k \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 2\cdot 5^k - 3^k & 5^k - 3^k \\ 2\cdot 3^k - 2\cdot 5^k & 2\cdot 3^k - 5^k \end{bmatrix} \end{aligned}$$

Diagonalization

DEFINITION: Let $A \in \mathcal{R}^{n \times n}$. A is **diagonalizable** $\iff$ $A = P\,D\,P^{-1}$ for an invertible matrix P and diagonal matrix D.

Thm: A is **diagonalizable** $\iff$ A has n linearly independent eigenvectors.

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Proof:

$$
\begin{aligned}
& A \text{ has } n \text{ L.I.D. eigenvectors} \\
\iff \quad & A\,\mathbf{v}_j = \lambda_j\,\mathbf{v}_j, \quad j = 1, 2, \cdots, n. \quad \mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n \text{ L.I.D.} \\
\iff \quad & A\left(\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\right) = \left(\lambda_1\,\mathbf{v}_1, \lambda_2\,\mathbf{v}_2, \cdots, \lambda_n\,\mathbf{v}_n\right), \\
& \mathbf{v}_j, \quad \mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n \text{ L.I.D.}. \quad (\ell)
\end{aligned}
$$

Thm: $A \in \mathbb{R}^n$ is similar to a diagonal matrix D if and only if A has n linearly independent eigenvectors.

Proof:

$$\begin{aligned} & A \text{ has } n \text{ L.I.D. eigenvectors} \\ \Longleftrightarrow \quad & A\,\mathbf{v}_j = \lambda_j\,\mathbf{v}_j, \quad j = 1, 2, \cdots, n. \quad \mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n \text{ L.I.D.} \\ \Longleftrightarrow \quad & A\left(\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\right) = \left(\lambda_1\,\mathbf{v}_1, \lambda_2\,\mathbf{v}_2, \cdots, \lambda_n\,\mathbf{v}_n\right), \\ & \mathbf{v}_j, \quad \mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n \text{ L.I.D..} \quad (\ell) \end{aligned}$$

Let $P = \left(\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\right)$, $D = \mathbf{diag}\left(\lambda_1, \lambda_2, \cdots, \lambda_n\right)$. Since $\left(\lambda_1\,\mathbf{v}_1, \lambda_2\,\mathbf{v}_2, \cdots, \lambda_n\,\mathbf{v}_n\right) = P\,D$. By (ℓ),

Thm: $A \in \mathbb{R}^n$ is similar to a diagonal matrix D if and only if A has n linearly independent eigenvectors.

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$$\begin{aligned} & A \text{ has } n \text{ L.I.D. eigenvectors} \\ \iff & A\,\mathbf{v}_j = \lambda_j\,\mathbf{v}_j, \quad j = 1, 2, \cdots, n. \quad \mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n \text{ L.I.D.} \\ \iff & A\left(\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\right) = \left(\lambda_1\,\mathbf{v}_1, \lambda_2\,\mathbf{v}_2, \cdots, \lambda_n\,\mathbf{v}_n\right), \\ & \mathbf{v}_j, \quad \mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n \text{ L.I.D..} \quad (\ell) \end{aligned}$$

Let $P = \left(\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\right)$, $D = \mathbf{diag}\left(\lambda_1, \lambda_2, \cdots, \lambda_n\right)$. Since $\left(\lambda_1\,\mathbf{v}_1, \lambda_2\,\mathbf{v}_2, \cdots, \lambda_n\,\mathbf{v}_n\right) = P\,D$. By (ℓ),

$$\begin{aligned} & A \text{ has } n \text{ L.I.D. eigenvectors} \\ \iff & A\,P = P\,D \quad P \text{ is invertible} \\ \iff & A = P\,D\,P^{-1} \end{aligned}$$

Thm: $A \in \mathbb{R}^n$ is similar to a diagonal matrix D if and only if A has n linearly independent eigenvectors.

Proof:

$$\begin{aligned} & A \text{ has } n \text{ L.I.D. eigenvectors} \\ \Longleftrightarrow \quad & A\,\mathbf{v}_j = \lambda_j\,\mathbf{v}_j, \quad j = 1, 2, \cdots, n. \quad \mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n \text{ L.I.D.} \\ \Longleftrightarrow \quad & A\left(\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\right) = \left(\lambda_1\,\mathbf{v}_1, \lambda_2\,\mathbf{v}_2, \cdots, \lambda_n\,\mathbf{v}_n\right), \\ & \mathbf{v}_j, \quad \mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n \text{ L.I.D..} \quad (\ell) \end{aligned}$$

Let $P = \left(\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\right)$, $D = \mathbf{diag}\left(\lambda_1, \lambda_2, \cdots, \lambda_n\right)$. Since $\left(\lambda_1\,\mathbf{v}_1, \lambda_2\,\mathbf{v}_2, \cdots, \lambda_n\,\mathbf{v}_n\right) = P\,D$. By (ℓ),

$$\begin{aligned} & A \text{ has } n \text{ L.I.D. eigenvectors} \\ \Longleftrightarrow \quad & A\,P = P\,D \quad P \text{ is invertible} \\ \Longleftrightarrow \quad & A = P\,D\,P^{-1} \end{aligned}$$

Cor: $A \in \mathbb{R}^n$ with n distinct eigenvalues is similar to diagonal matrix.

EX: Diagonalize the matrix $A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$.

SOLUTION:

- FIND E-VALUES:

$$\mathbf{det}\,(A - \lambda I) = \mathbf{det}\begin{pmatrix} 1-\lambda & 3 & 3 \\ -3 & -5-\lambda & -3 \\ 3 & 3 & 1-\lambda \end{pmatrix}$$

$$= \quad (1-\lambda)\,\mathbf{det}\begin{pmatrix} \square & \square & \square \\ \square & -5-\lambda & -3 \\ \square & 3 & 1-\lambda \end{pmatrix} - (-3)\,\mathbf{det}\begin{pmatrix} \square & 3 & 3 \\ \square & \square & \square \\ \square & 3 & 1-\lambda \end{pmatrix}$$

$$+\,(3)\,\mathbf{det}\begin{pmatrix} \square & 3 & 3 \\ \square & -5-\lambda & -3 \\ \square & \square & \square \end{pmatrix}$$

$$= \quad (1-\lambda)\,(2+\lambda)^2$$

Eigenvalues are $\lambda_1 = 1, \lambda_2 = \lambda_3 = -2$.

EX: Diagonalize the matrix $A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$.

SOLUTION: Eigenvalues are $\lambda_1 = 1, \lambda_2 = \lambda_3 = -2$.

- FIND E-VECTORS:

$$(A - \lambda_1 I)\, \mathbf{v}_1 = \mathbf{0}, \quad \Longrightarrow \quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix},$$

$$(A - \lambda_2 I)\, \mathbf{v} = \mathbf{0}, \begin{bmatrix} 3 & 3 & 3 \\ -3 & -3 & -3 \\ 3 & 3 & 3 \end{bmatrix} \mathbf{v} = \mathbf{0}, \Longrightarrow \mathbf{v} = v_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + v_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix},$$

$$\Longrightarrow \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

EX: Diagonalize the matrix $A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$.

SOLUTION:

- Eigenvalues are $\lambda_1 = 1, \lambda_2 = \lambda_3 = -2$ with eigenvectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

Define: $P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}.$

Then $P^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$

EX: Diagonalize the matrix $A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$.

SOLUTION:

- Eigenvalues are $\lambda_1 = 1, \lambda_2 = \lambda_3 = -2$ with eigenvectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

Define: $P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$.

$$A\,P = A \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix} = \begin{bmatrix} A\mathbf{v}_1 & A\mathbf{v}_2 & A\mathbf{v}_3 \end{bmatrix} = \begin{bmatrix} \lambda_1\mathbf{v}_1 & \lambda_2\mathbf{v}_2 & \lambda_3\mathbf{v}_3 \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = P\,D, \quad \Longrightarrow \quad A = P\,D\,P^{-1}.$$

EX: Diagonalize $A = \begin{bmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{bmatrix}$, IF POSSIBLE

SOLUTION:

- FIND E-VALUES:

$$\textbf{det}\,(A - \lambda\, I) = \textbf{det} \begin{pmatrix} 2-\lambda & 4 & 3 \\ -4 & -6-\lambda & -3 \\ 3 & 3 & 1-\lambda \end{pmatrix}$$

$$= \quad (2-\lambda)\ \textbf{det} \begin{pmatrix} \square & \square & \square \\ \square & -6-\lambda & -3 \\ \square & 3 & 1-\lambda \end{pmatrix} - (-4)\ \textbf{det} \begin{pmatrix} \square & 4 & 3 \\ \square & \square & \square \\ \square & 3 & 1-\lambda \end{pmatrix}$$

$$+ (3)\ \textbf{det} \begin{pmatrix} \square & 4 & 3 \\ \square & -6-\lambda & -3 \\ \square & \square & \square \end{pmatrix}$$

$$= \quad (1-\lambda)\,(2+\lambda)^2$$

Eigenvalues are $\lambda_1 = 1, \lambda_2 = \lambda_3 = -2.$

EX: Diagonalize $A = \begin{bmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{bmatrix}$, IF POSSIBLE

SOLUTION: Eigenvalues are $\lambda_1 = 1, \lambda_2 = \lambda_3 = -2$.

- FIND E-VECTORS:

$$(A - \lambda_1 I)\,\mathbf{v}_1 = \mathbf{0}, \quad \Longrightarrow \quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix},$$

$$(A - \lambda_2 I)\,\mathbf{v} = \mathbf{0}, \quad \begin{bmatrix} 4 & 4 & 3 \\ -4 & -4 & -3 \\ 3 & 3 & 3 \end{bmatrix} \mathbf{v} = \mathbf{0}, \quad \Longrightarrow \quad \mathbf{v} = v_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix},$$

$$\Longrightarrow \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \mathbf{?}.$$

$\#$ of L.I.D. eigenvectors $<$ matrix dimension $\Longleftrightarrow$ NO DIAGONALIZATION