Dimension of Vector Space: **Theorem** 11

LET: H be a subspace of finite-dimensional vector space V.

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(e) Repeat steps **(a-c)** to continue expanding $\mathcal{S}$ until it spans H and hence is a basis. **QED**

Dimension of Vector Space: **Theorem** 12

Let $V \neq \{\mathbf{0}\}$ be p-dimensional vector space, $\mathcal{S} = \{\mathbf{u}_1, \cdots, \mathbf{u}_p\} \subseteq H$. Then

$$\mathcal{S} \text{ is basis} \iff \mathcal{S} \text{ spans } V \iff \text{Vectors in } \mathcal{S} \text{ linearly independent.}$$

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- Since $\mathcal{S}$ spans V, a subset of $\mathcal{S}$ must be basis for V (**Thm 5**.)
- $p = \dim V = \#$ of vectors in the subset. (by DEF.)
- Subset must be $\mathcal{S}$. **QED**

$$
\begin{aligned}
A &= \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 & \mathbf{a}_5 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 4 & 0 & 2 & -1 \\ 3 & 12 & 1 & 5 & 5 \\ 2 & 8 & 1 & 3 & 2 \\ 5 & 20 & 2 & 8 & 8 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & 4 & 0 & 2 & -1 \\ 0 & 0 & \boxed{1} & -1 & 8 \\ 0 & 0 & 0 & 0 & \boxed{\text{-4}} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.
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- Pivot columns are columns $1, 3, 5$: Col $A = \textbf{Span}\,(\mathbf{a}_1, \mathbf{a}_3, \mathbf{a}_5)$.

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$$\mathbf{x} = x_2 \begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}. \implies \text{Nul } A = \textbf{Span}\left(\begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right).$$

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$$\dim(\text{Col } A) = 3 = \# \text{ of pivots},$$
$$\dim(\text{Nul } A) = 2 = \# \text{ of free variables}.$$

§4.6 Rank

DEF: Let $A \in \mathcal{R}^{m\times n}$. Then Row $A \overset{def}{=}$ the **span** of row vectors of A.

EX: ROW VECTOR FORM A

$$A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix} \overset{def}{=} \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \mathbf{r}_3 \\ \mathbf{r}_4 \end{bmatrix}$$

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Let B be obtained from A with elementary operations $\Longrightarrow$ Row $A =$ Row B.

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$$\boxed{\mathbf{dim}\left(\text{Col } A\right) + \mathbf{dim}\left(\text{Nul } A\right) = \#\text{ of columns}}$$

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The Invertible Matrix Theorem (continued)

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Let $A \in \mathcal{R}^{n \times n}$. Then following statements are equivalent.

a. A is an invertible matrix.

m. Columns of A form a basis for $\mathcal{R}^n$.

p. $\textbf{rank}\,(A) = n$.

r. $\textbf{dim}\,(\text{Nul } A) = 0$.

§4.7 Change of Basis

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Let $\mathbf{x} \in \mathcal{R}^n$, let $\mathcal{B}$ and $\mathcal{C}$ be two bases for $\mathcal{R}^n$.

- $[\mathbf{x}]_{\mathcal{B}}$ and $[\mathbf{x}]_{\mathcal{C}}$ are coordinates in bases $\mathcal{B}$ and $\mathcal{C}$, respectively.
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EX: Let $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, $[\mathbf{x}]_{\mathcal{C}} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$. Then $[\mathbf{x}]_{\mathcal{C}} = [?] \cdot [\mathbf{x}]_{\mathcal{B}}$

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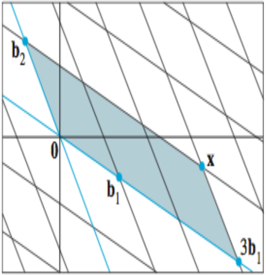

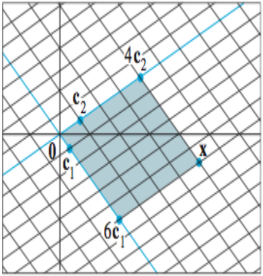

Change of Basis: **Example**

Consider two <u>bases</u> for vector space V:

$$\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}, \quad \mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}.$$

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$$\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}, \quad \mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}.$$

$$\text{with} \quad \mathbf{b}_1 = 4\,\mathbf{c}_1 + \mathbf{c}_2, \quad \text{and} \quad \mathbf{b}_2 = -6\,\mathbf{c}_1 + \mathbf{c}_2.$$

Suppose $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, find $[\mathbf{x}]_{\mathcal{C}}$.

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SOLUTION: Basis relations are

$$[\mathbf{b}_1]_{\mathcal{C}} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}, \quad [\mathbf{b}_2]_{\mathcal{C}} = \begin{bmatrix} -6 \\ 1 \end{bmatrix}.$$

$$\begin{aligned} [\mathbf{x}]_{\mathcal{C}} &= [3\,\mathbf{b}_1 + \mathbf{b}_2]_{\mathcal{C}} = 3\,[\mathbf{b}_1]_{\mathcal{C}} + [\mathbf{b}_2]_{\mathcal{C}} \\ &= [[\mathbf{b}_1]_{\mathcal{C}}\,, [\mathbf{b}_2]_{\mathcal{C}}] \begin{bmatrix} 3 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 4 & -6 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix} \end{aligned}$$

Change of Basis

Consider two bases for vector space V:

$$\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_n\}, \quad \mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2, \cdots, \mathbf{c}_n\}.$$

Change of Basis

Consider two bases for vector space V:

$$\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_n\}, \quad \mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2, \cdots, \mathbf{c}_n\}.$$

change-of-coordinates matrix: $\mathcal{C} \overset{\mathbf{P}}{\leftarrow} \mathcal{B} \overset{def}{=} [[\mathbf{b}_1]_{\mathcal{C}}, [\mathbf{b}_2]_{\mathcal{C}}, \cdots, [\mathbf{b}_n]_{\mathcal{C}}].$

Change of Basis

Consider two bases for vector space V:

$$\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_n\}, \quad \mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2, \cdots, \mathbf{c}_n\}.$$

change-of-coordinates matrix: $\underset{\mathcal{C} \leftarrow \mathcal{B}}{\mathbf{P}} \overset{def}{=} \left[[\mathbf{b}_1]_{\mathcal{C}}, [\mathbf{b}_2]_{\mathcal{C}}, \cdots, [\mathbf{b}_n]_{\mathcal{C}}\right].$

Thm: For any vector $\mathbf{x} \in V$, $[\mathbf{x}]_{\mathcal{C}} = \underset{\mathcal{C} \leftarrow \mathcal{B}}{\mathbf{P}} \; [\mathbf{x}]_{\mathcal{B}}$

Change of Basis

Consider two bases for vector space V:

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change-of-coordinates matrix: $\underset{\mathcal{C} \leftarrow \mathcal{B}}{\mathbf{P}} \overset{def}{=} [[\mathbf{b}_1]_{\mathcal{C}}, [\mathbf{b}_2]_{\mathcal{C}}, \cdots, [\mathbf{b}_n]_{\mathcal{C}}]$.

Thm: For any vector $\mathbf{x} \in V$, $[\mathbf{x}]_{\mathcal{C}} = \underset{\mathcal{C} \leftarrow \mathcal{B}}{\mathbf{P}} [\mathbf{x}]_{\mathcal{B}}$

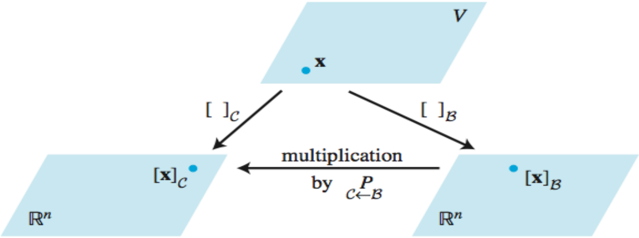

FIGURE 2 Two coordinate systems for V.

Change of Basis

Consider two bases for vector space V:

$$\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_n\}, \quad \mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2, \cdots, \mathbf{c}_n\}.$$

Change of Basis

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Change of Basis

Consider two bases for vector space V:

$$\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_n\}, \quad \mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2, \cdots, \mathbf{c}_n\}.$$

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Thm: For any vector $\mathbf{x} \in V$, $[\mathbf{x}]_{\mathcal{C}} = \underset{\mathcal{C} \leftarrow \mathcal{B}}{\mathbf{P}} [\mathbf{x}]_{\mathcal{B}}$

PROOF: Let $\mathbf{x} = \alpha_1 \mathbf{b}_1 + \cdots + \alpha_n \mathbf{b}_n$ so that $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}.$

Change of Basis

Consider two bases for vector space V:

$$\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_n\}, \quad \mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2, \cdots, \mathbf{c}_n\}.$$

change-of-coordinates matrix: $\underset{\mathcal{C} \leftarrow \mathcal{B}}{\mathbf{P}} \stackrel{def}{=} [[\mathbf{b}_1]_{\mathcal{C}}, [\mathbf{b}_2]_{\mathcal{C}}, \cdots, [\mathbf{b}_n]_{\mathcal{C}}]$.

Thm: For any vector $\mathbf{x} \in V$, $[\mathbf{x}]_{\mathcal{C}} = \underset{\mathcal{C} \leftarrow \mathcal{B}}{\mathbf{P}} [\mathbf{x}]_{\mathcal{B}}$

PROOF: Let $\mathbf{x} = \alpha_1 \mathbf{b}_1 + \cdots + \alpha_n \mathbf{b}_n$ so that $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}$.

$$[\mathbf{x}]_{\mathcal{C}} = [\alpha_1 \mathbf{b}_1 + \cdots + \alpha_n \mathbf{b}_n]_{\mathcal{C}}$$

Change of Basis

Consider two <u>bases</u> for vector space V:

$$\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_n\}, \quad \mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2, \cdots, \mathbf{c}_n\}.$$

change-of-coordinates matrix: $\underset{\mathcal{C} \leftarrow \mathcal{B}}{\mathbf{P}} \stackrel{def}{=} \left[[\mathbf{b}_1]_{\mathcal{C}}, [\mathbf{b}_2]_{\mathcal{C}}, \cdots, [\mathbf{b}_n]_{\mathcal{C}}\right].$

Thm: For any vector $\mathbf{x} \in V$, $[\mathbf{x}]_{\mathcal{C}} = \underset{\mathcal{C} \leftarrow \mathcal{B}}{\mathbf{P}} \; [\mathbf{x}]_{\mathcal{B}}$

PROOF: Let $\mathbf{x} = \alpha_1 \mathbf{b}_1 + \cdots + \alpha_n \mathbf{b}_n$ so that $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}$.

$$[\mathbf{x}]_{\mathcal{C}} = [\alpha_1 \mathbf{b}_1 + \cdots + \alpha_n \mathbf{b}_n]_{\mathcal{C}} = \alpha_1 [\mathbf{b}_1]_{\mathcal{C}} + \cdots + \alpha_n [\mathbf{b}_n]_{\mathcal{C}}$$

Change of Basis

Consider two bases for vector space V:

$$\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_n\}, \quad \mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2, \cdots, \mathbf{c}_n\}.$$

change-of-coordinates matrix: $\underset{\mathcal{C} \leftarrow \mathcal{B}}{\mathbf{P}} \overset{def}{=} \left[[\mathbf{b}_1]_{\mathcal{C}}, [\mathbf{b}_2]_{\mathcal{C}}, \cdots, [\mathbf{b}_n]_{\mathcal{C}} \right].$

Thm: For any vector $\mathbf{x} \in V$, $[\mathbf{x}]_{\mathcal{C}} = \underset{\mathcal{C} \leftarrow \mathcal{B}}{\mathbf{P}} \; [\mathbf{x}]_{\mathcal{B}}$

PROOF: Let $\mathbf{x} = \alpha_1 \mathbf{b}_1 + \cdots + \alpha_n \mathbf{b}_n$ so that $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}$.

$$\begin{aligned} [\mathbf{x}]_{\mathcal{C}} &= [\alpha_1 \mathbf{b}_1 + \cdots + \alpha_n \mathbf{b}_n]_{\mathcal{C}} = \alpha_1 [\mathbf{b}_1]_{\mathcal{C}} + \cdots + \alpha_n [\mathbf{b}_n]_{\mathcal{C}} \\ &= \left[[\mathbf{b}_1]_{\mathcal{C}}, [\mathbf{b}_2]_{\mathcal{C}}, \cdots, [\mathbf{b}_n]_{\mathcal{C}} \right] \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} \end{aligned}$$

Change of Basis

Consider two bases for vector space V:

$$\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_n\}, \quad \mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2, \cdots, \mathbf{c}_n\}.$$

change-of-coordinates matrix: $\underset{\mathcal{C} \leftarrow \mathcal{B}}{\mathbf{P}} \overset{def}{=} \left[[\mathbf{b}_1]_{\mathcal{C}}, [\mathbf{b}_2]_{\mathcal{C}}, \cdots, [\mathbf{b}_n]_{\mathcal{C}}\right].$

Thm: For any vector $\mathbf{x} \in V$, $[\mathbf{x}]_{\mathcal{C}} = \underset{\mathcal{C} \leftarrow \mathcal{B}}{\mathbf{P}} \; [\mathbf{x}]_{\mathcal{B}}$

PROOF: Let $\mathbf{x} = \alpha_1 \mathbf{b}_1 + \cdots + \alpha_n \mathbf{b}_n$ so that $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}.$

$$\begin{aligned} [\mathbf{x}]_{\mathcal{C}} &= [\alpha_1 \mathbf{b}_1 + \cdots + \alpha_n \mathbf{b}_n]_{\mathcal{C}} = \alpha_1 [\mathbf{b}_1]_{\mathcal{C}} + \cdots + \alpha_n [\mathbf{b}_n]_{\mathcal{C}} \\ &= \left[[\mathbf{b}_1]_{\mathcal{C}}, [\mathbf{b}_2]_{\mathcal{C}}, \cdots, [\mathbf{b}_n]_{\mathcal{C}}\right] \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} = \underset{\mathcal{C} \leftarrow \mathcal{B}}{\mathbf{P}} \; [\mathbf{x}]_{\mathcal{B}} \end{aligned}$$

EX: Consider two bases for $\mathcal{R}^2$: $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$, $\quad \mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$.

EX: Consider two bases for $\mathcal{R}^2$: $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$, $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$.

with $\mathbf{b}_1 = \begin{bmatrix} -9 \\ 1 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} -5 \\ -1 \end{bmatrix}$, $\mathbf{c}_1 = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$, $\mathbf{c}_2 = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$.

Find $\underset{\mathcal{C} \leftarrow \mathcal{B}}{\mathbf{P}} = [[\mathbf{b}_1]_{\mathcal{C}}, [\mathbf{b}_2]_{\mathcal{C}}]$

EX: Consider two bases for $\mathcal{R}^2$: $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$, $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$.

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Find $\underset{\mathcal{C} \leftarrow \mathcal{B}}{\mathbf{P}} = [[\mathbf{b}_1]_{\mathcal{C}}, [\mathbf{b}_2]_{\mathcal{C}}]$

SOLUTION: Let $[\mathbf{b}_1]_{\mathcal{C}} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$. Then

$\mathbf{b}_1 = \alpha_1 \mathbf{c}_1 + \alpha_2 \mathbf{c}_2$

EX: Consider two bases for $\mathcal{R}^2$: $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$, $\quad \mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$.

with $\quad \mathbf{b}_1 = \begin{bmatrix} -9 \\ 1 \end{bmatrix}, \ \mathbf{b}_2 = \begin{bmatrix} -5 \\ -1 \end{bmatrix}, \ \mathbf{c}_1 = \begin{bmatrix} 1 \\ -4 \end{bmatrix}, \ \mathbf{c}_2 = \begin{bmatrix} 3 \\ -5 \end{bmatrix}.$

Find $\quad \underset{\mathcal{C} \leftarrow \mathcal{B}}{\mathbf{P}} = [[\mathbf{b}_1]_{\mathcal{C}}, [\mathbf{b}_2]_{\mathcal{C}}]$

SOLUTION: Let $[\mathbf{b}_1]_{\mathcal{C}} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$. Then

$$\mathbf{b}_1 = \alpha_1 \, \mathbf{c}_1 + \alpha_2 \, \mathbf{c}_2 = \begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$$

EX: Consider two bases for $\mathcal{R}^2$: $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$, $\quad \mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$.

with $\quad \mathbf{b}_1 = \begin{bmatrix} -9 \\ 1 \end{bmatrix}, \ \mathbf{b}_2 = \begin{bmatrix} -5 \\ -1 \end{bmatrix}, \ \mathbf{c}_1 = \begin{bmatrix} 1 \\ -4 \end{bmatrix}, \ \mathbf{c}_2 = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$.

Find $\quad \mathcal{C} \overset{\mathbf{P}}{\leftarrow} \mathcal{B} \ = [[\mathbf{b}_1]_\mathcal{C}, [\mathbf{b}_2]_\mathcal{C}]$

SOLUTION: Let $[\mathbf{b}_1]_\mathcal{C} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$. Then

$$\mathbf{b}_1 = \alpha_1 \mathbf{c}_1 + \alpha_2 \mathbf{c}_2 = \begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \Longrightarrow \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 \end{bmatrix}^{-1} \mathbf{b}_1.$$

therefore $\ [\mathbf{b}_1]_\mathcal{C} = \begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 \end{bmatrix}^{-1} \mathbf{b}_1$, and $\ [\mathbf{b}_2]_\mathcal{C} = \begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 \end{bmatrix}^{-1} \mathbf{b}_2$.

$$\begin{aligned} \mathcal{C} \overset{\mathbf{P}}{\leftarrow} \mathcal{B} &= [[\mathbf{b}_1]_\mathcal{C}, [\mathbf{b}_2]_\mathcal{C}] = \begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 3 \\ -4 & -5 \end{bmatrix}^{-1} \begin{bmatrix} -9 & -5 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 6 & 4 \\ -5 & -3 \end{bmatrix} \end{aligned}$$