§3.3 Determinant as Area

Thm: For $A \in \mathcal{R}^{2\times 2}$, the area of the parallelogram determined by the columns of A is $|\mathbf{det}(A)|$.

PROOF:

- If A is not invertible, then $|\mathbf{det}(A)| = 0$. Columns of A are parallel, hence parallelogram becomes a line segment, with area $= 0$.

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- We now assume If A is invertible in the rest of the proof.

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PROOF: If $A = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$ is diagonal. Then $|\mathbf{det}\,(A)| = |a\,d|$.

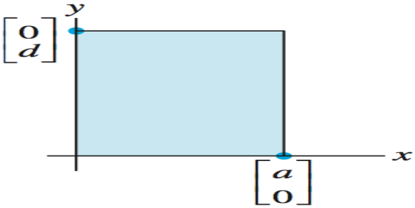

FIGURE 1
Area $= |a\,d|$.

Area of parallelogram is also $|a\,d|$.

§3.3 Determinant as Area

Thm: For $A \in \mathcal{R}^{2\times 2}$, the volume of the parallelogram determined by the columns of A is $|\mathbf{det}\,(A)|$.

PROOF: Let $A \in \mathcal{R}^{2\times 2}$ be invertible. A can be reduced to diagonal matrix with two types of operations:

- interchange two columns.
  This operation does not change $|\mathbf{det}\,(A)|$ or area of the parallelogram.

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- **one row** $+\, c \times$ **another** $\Longrightarrow$ **same row**
  This operation does not change $|\mathbf{det}\,(A)|$.
  - Now only need to prove this operation does not change area either.

Let $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 \end{bmatrix}$, and $B = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 + c\,\mathbf{a}_1 \end{bmatrix}$. Then $|\mathbf{det}\,(A)| = |\mathbf{det}\,(B)|$

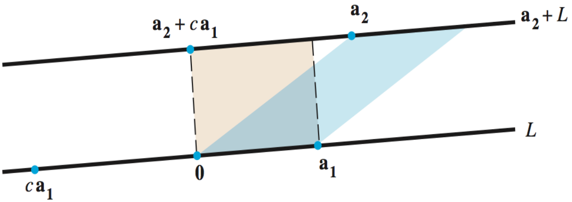

FIGURE 2 Two parallelograms of equal area.

- L is a line through $\mathbf{0}$ and $\mathbf{a}_1$.
- $\mathbf{a}_2 + L$ is a line through $\mathbf{a}_2$ and parallel to L.
- Both parallelograms have same base and height, hence same area.

§3.3 Determinant as Volume

Thm: For $A \in \mathcal{R}^{3\times3}$, the volume of the parallelepiped determined by the columns of A is $|\mathbf{det}\,(A)|$.

PROOF: If $A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$ is diagonal. Then

$|\mathbf{det}\,(A)| = |a\,b\,c|$.

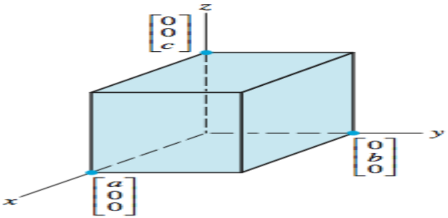

FIGURE 3
Volume $= |abc|$.

Volume of parallelepiped is also $|a\,b\,c|$.

§3.3 Determinant as Volume

Thm: For $A \in \mathcal{R}^{3\times 3}$, the volume of the parallelepiped determined by the columns of A is $|\mathbf{det}\,(A)|$.

PROOF: Let $A \in \mathcal{R}^{3\times 3}$ be invertible. A can be reduced to diagonal matrix with two types of operations:

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Let $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{bmatrix}$, and $B = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 + c\,\mathbf{a}_1 & \mathbf{a}_3 \end{bmatrix}$. Then $|\mathbf{det}\,(A)| = |\mathbf{det}\,(B)|$

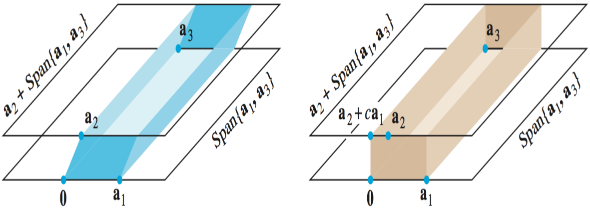

FIGURE 4 Two parallelepipeds of equal volume.

- Base in **Span** $(\mathbf{a}_1, \mathbf{a}_3)$.
- $\mathbf{a}_2 + \mathbf{Span}\,(\mathbf{a}_1, \mathbf{a}_3)$ is a plane parallel $\mathbf{Span}\,(\mathbf{a}_1, \mathbf{a}_3)$.
- Both parallelepipeds have same base and height, hence same volume.

Vector Space and Subspace

- **Vector Space** is a set V of objects (vectors)
- OPERATIONS: *addition* and *scalar multiplication*
- Axioms below work for all $\mathbf{u}, \mathbf{v} \in V$ and all scalars.

1. The sum of $\mathbf{u}$ and $\mathbf{v}$, denoted by $\mathbf{u} + \mathbf{v}$, is in V.
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
3. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$.
4. There is a **zero** vector $\mathbf{0}$ in V such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$.
5. For each $\mathbf{u}$ in V, there is a vector $-\mathbf{u}$ in V such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.
6. The scalar multiple of $\mathbf{u}$ by c, denoted by $c\mathbf{u}$, is in V.
7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$.
8. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$.
9. $c(d\mathbf{u}) = (cd)\mathbf{u}$.
10. $1\mathbf{u} = \mathbf{u}$.

Vector Space: Example (I)

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$V = \mathcal{R}^n$.

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$V = \mathcal{P}_3$, set of all polynomials of degree at most 3, of form:

$$\mathbf{p}(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3.$$

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- ADDITION: Let $\mathbf{q}\,(t) = b_0 + b_1\,t + b_2\,t^2 + b_3 t^3$.

$$(\mathbf{p} + \mathbf{q})\,(t) \stackrel{def}{=} (a_0 + b_0) + (a_1 + b_1)\ t + (a_2 + b_2)\ t^2 + (a_3 + b_3)\,t^3 \in \mathcal{P}_3$$

- SCALAR MULTIPLICATION:

$$(\alpha\,\mathbf{p})\,(t) \stackrel{def}{=} (\alpha\,a_0) + (\alpha\,a_1)\ t + (\alpha\,a_2)\ t^2 + (\alpha\,a_3)\,t^3 \in \mathcal{P}_3$$

Vector Space: Example (III)

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$\mathcal{C}$ = set of all continuous functions:

Sums, scalar multiples of continuous functions are continuous functions.

Vector Space is a set

If needles were vectors, then the cactus would be vector space.

Vectors can point to all possible directions, have all possible sizes.

Subspace


May 5, 2018
SUBSPACE
COMICS
276
Subspace Comics @subspacecomics · 2h
We are closed today as nature feels we need a day off. Normal hours shall

Subspace

Subspace H is a Vector Space

Subspace

Subspace H is a Vector Space that is subset of a Vector Space V: $H \subseteq V$.

Subspace: Example (I)

- $H = \mathcal{P}_3$, set of all polynomials of degree at most 3, of form:

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Both H and V are vector spaces, with $H \subset V$. So H is a subspace.

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Axioms 1.) and 6.) need to work for H

Subspace is Vector Space (II)

Let $H \subseteq V$. Let V be vector space. H is a subspace of V if

- $\mathbf{0} \in H$.
- For any scalar c and any $\mathbf{u} \in H$, $\Longrightarrow c\,\mathbf{u} \in H$.
- For any $\mathbf{u}, \mathbf{v} \in H$, $\Longrightarrow \mathbf{u} + \mathbf{v} \in H$.

Subspace Example (II): solutions to homogeneous equations

Let $A \in \mathcal{R}^{m \times n}$, and let H be set of all solutions to $A\mathbf{x} = \mathbf{0}$.

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So H is a subspace of $\mathcal{C}$: set of all continuous functions.

Subspace

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SKETCHY PROOF: Let $\mathbf{u}, \mathbf{v} \in$ **Span** $(\mathbf{v}_1, \cdots, \mathbf{v}_p)$.
Then $\mathbf{u}, \mathbf{v}$ are linear combinations of $\mathbf{v}_1, \cdots, \mathbf{v}_p$,
so are $c\,\mathbf{u}, \mathbf{u} + \mathbf{v}$.
Hence $c\,\mathbf{u}, \mathbf{u} + \mathbf{v} \in$ **Span** $(\mathbf{v}_1, \cdots, \mathbf{v}_p)$. **QED**

Subspace Example (IV): solutions to differential equation

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Let H be set of solutions to differential equation:

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- Two linearly independent solutions to (ℓ):

$$y_1(t) = \mathbf{sin}(t), \quad y_2(t) = \mathbf{cos}(t).$$

- Solution subspace

$$H = \mathbf{Span}(\mathbf{sin}(t), \mathbf{cos}(t)).$$

§4.2 Null Spaces, Column Spaces

Let $A = \left[\begin{array}{ccc} \mathbf{a}_1, & \cdots, & \mathbf{a}_n \end{array} \right] \in \mathcal{R}^{m \times n}$.

- The **null space** of A, denoted Nul A, is set of solutions to $A\mathbf{x} = \mathbf{0}$:

$$\text{Nul } A \stackrel{def}{=} \{\mathbf{x} \mid \mathbf{x} \in \mathcal{R}^n \text{ and } A\mathbf{x} = \mathbf{0}\}.$$

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- The **column space** of A, denoted Col A, is set of linear combinations of $\mathbf{a}_1, \cdots, \mathbf{a}_n$:

$$\text{Col } A \overset{def}{=} \textbf{Span}\left(\mathbf{a}_1, \cdots, \mathbf{a}_n\right).$$

Example: Null Spaces, Column Spaces

Let $A = \begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1, & \mathbf{a}_2, & \mathbf{a}_3 \end{bmatrix} \in \mathcal{R}^{2\times 3}$.

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- row echelon form: $A \sim \begin{bmatrix} 1 & -3 & -2 \\ 0 & -6 & -9 \end{bmatrix}$ with free variable x_3.

$$\text{Solutions to } A\mathbf{x} = \mathbf{0}: \quad \mathbf{x} \; = \; -\frac{x_3}{2}\begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix}.$$

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$$= \textbf{Span}\left(\begin{bmatrix} 1 \\ -5 \end{bmatrix}, \begin{bmatrix} -3 \\ 9 \end{bmatrix} \right).$$

Thm: Null Space is a subspace

PROOF: Let $A \in \mathcal{R}^{m \times n}$. Want to show Nul A is a subspace.

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- For any $\mathbf{u}, \mathbf{v} \in$ Nul A, $\Longrightarrow A\,\mathbf{u} = A\,\mathbf{v} = \mathbf{0}; A\,(\mathbf{u} + \mathbf{v}) = \mathbf{0}$.
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  $\Longrightarrow A\,\mathbf{u} = \mathbf{0}; A\,(c\,\mathbf{u}) = \mathbf{0}$. So $c\,\mathbf{u} \in$ Nul A.
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  So $\mathbf{u} + \mathbf{v} \in$ Nul A.

Hence Nul A is a subspace of $\mathcal{R}^n$.

Kernel and Range of Linear Transform

DEF: A **linear transformation** T from vector space V into vector W is a rule: $\mathbf{x} \in V \mapsto T(\mathbf{x}) \in W$, such that

$$\begin{aligned} T(\mathbf{u}+\mathbf{v}) &= T(\mathbf{u}) + T(\mathbf{v}), \quad \text{for all} \quad \mathbf{u}, \mathbf{v} \in V \\ T(c\,\mathbf{u}) &= c\,T(\mathbf{u}), \quad \text{for all} \quad \mathbf{u} \quad \text{and for all scalar } c \end{aligned}$$

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EX: Let V be set of all second-order differentiable functions,

$$T(y) = y'' + y, \quad \text{where} \quad y = y(t) \in V.$$

T is a linear transformation.

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Let $T : \mathbf{x} \in V \mapsto T(\mathbf{x}) \in W$ be a linear transformation.

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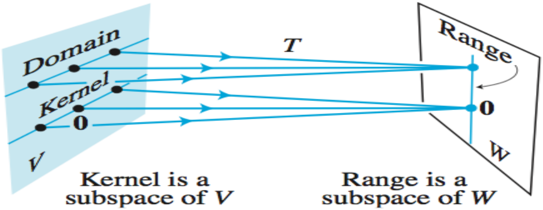

FIGURE 2 Subspaces associated with a linear transformation.

§4.3 Linearly independent sets; Bases

Let $\mathcal{S} \stackrel{def}{=} \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_p\}$ be a set of vectors in vector space V.

- $\mathcal{S}$ is **linearly dependent** if there exists a non-trivial solution to (ℓ):

$$\mathbf{v}_1\, c_1 + \mathbf{v}_2\, c_2 + \cdots + \mathbf{v}_p\, c_p = \mathbf{0}. \qquad (\ell)$$

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- $\mathcal{S}$ is **linearly independent** if there exists only trivial solution to (ℓ).
- Assume $p \geq 2$ and $\mathbf{v}_1 \neq \mathbf{0}$.
  $\mathcal{S}$ is linearly independent $\iff$
  some $\mathbf{v}_j$ (with $j > 1$) is linear combination of $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_{j-1}$.

Linearly independent sets: Examples

- The set $\{\mathbf{sin}(x), \mathbf{cos}(x), 1\}$ is **linearly independent**:

$$\mathbf{sin}(x)\, c_1 + \mathbf{cos}(x)\, c_2 + c_3 \equiv 0, \qquad \Longrightarrow c_1 = c_2 = c_3 = 0.$$

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Basis for Vector Space

Let H be a subspace of vector space V.

Indexed set of vectors $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_p\} \subset V$ is **basis** for H if

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EX: Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -4 \\ 1 \\ 7 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix}$. Then

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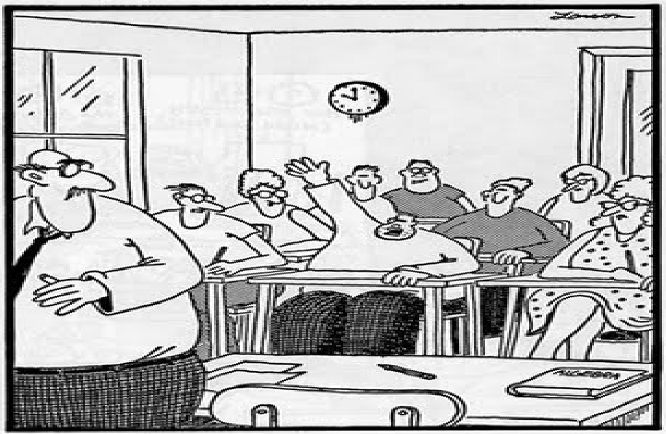
- Therefore must be basis for $\mathcal{R}^3$.

When vectors seem to span brain



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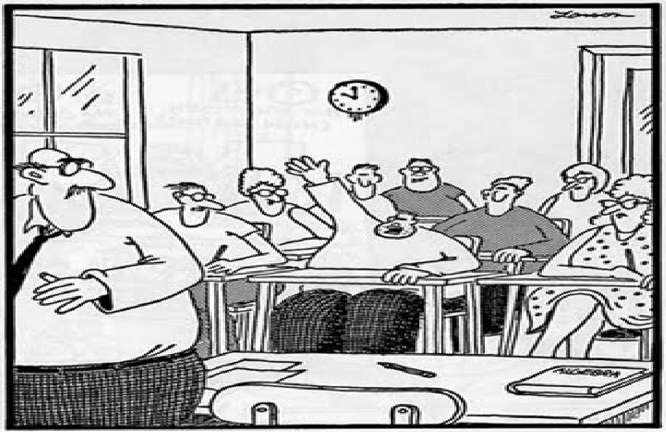

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But they don't

Spanning Set Theorem (I)

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$$\begin{aligned} \mathbf{x} &= c_1\,\mathbf{v}_1 + c_2\,\mathbf{v}_2 + c_3\,\mathbf{v}_3 \\ &= c_1\,\mathbf{v}_1 + c_2\,\mathbf{v}_2 + c_3\,(2\,\mathbf{v}_1 + \mathbf{v}_2) \\ &= (c_1 + 2\,c_3)\,\mathbf{v}_1 + (c_2 + c_3)\,\mathbf{v}_2 \in \mathbf{Span}(\mathbf{v}_1, \mathbf{v}_2) \quad \textbf{QED} \end{aligned}$$

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Let $\mathcal{S} = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_p\} \subset$ vector space V, $H = \mathbf{Span}\,(\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_p)$.

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since $\quad \mathbf{v}_k = a_1\,\mathbf{v}_1 + \cdots + a_{k-1}\,\mathbf{v}_{k-1} + a_{k+1}\,\mathbf{v}_{k+1} + \cdots + a_p\,\mathbf{v}_p, \quad (\ell) \quad$ becomes

$$\begin{aligned} \mathbf{x} \quad = \quad & (c_1 + c_k\,a_1)\,\mathbf{v}_1 + \cdots + (c_{k-1} + c_k\,a_{k-1})\,\mathbf{v}_{k-1} \\ & + (c_{k+1} + c_k\,a_{k+1})\,\mathbf{v}_{k+1} + \cdots + (c_p + c_k\,a_p)\,\mathbf{v}_p. \end{aligned}$$

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(c) Since $H \neq \{\mathbf{0}\}$, repeat steps **(a-b)** to continue reducing spanning set until it is linearly independent and hence a basis. **QED**

Example: Computing bases for Nul A and Col A

Let $$A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 & \mathbf{a}_5 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 0 & 2 & -1 \\ 3 & 12 & 1 & 5 & 5 \\ 2 & 8 & 1 & 3 & 2 \\ 5 & 20 & 2 & 8 & 8 \end{bmatrix}.$$

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- Compute $A \overset{echelon}{\Longrightarrow} \begin{bmatrix} \boxed{1} & 4 & 0 & 2 & -1 \\ 0 & 0 & \boxed{1} & -1 & 8 \\ 0 & 0 & 0 & 0 & \boxed{\text{-4}} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

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- Free columns are columns $2, 4$: Letting $A\mathbf{x} = \mathbf{0}$ gives

$$\mathbf{x} = x_2 \begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}. \implies \text{Nul } A = \textbf{Span}\left(\begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right).$$

§4.4 Coordinate Systems (I)

The Unique Representation Theorem

Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a basis for a vector space V. Then for each $\mathbf{x}$ in V, there exists a unique set of scalars $c_1, \ldots, c_n$ such that

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PROOF: Since $\mathcal{B}$ spans V, we write $\mathbf{x}$ as a linear combination

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for scalars $d_1, \cdots, d_n$. To show uniqueness, we now show equations (1) and (2) are same. Indeed,

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for scalars $d_1, \cdots, d_n$. To show uniqueness, we now show equations (1) and (2) are same. Indeed,

$$\mathbf{0} = \mathbf{x} - \mathbf{x} = (d_1 - c_1)\,\mathbf{b}_1 + \cdots + (d_n - c_n)\,\mathbf{b}_n.$$

Since $\mathcal{B}$ is a basis, vectors $\mathbf{b}_1, \cdots, \mathbf{b}_n$ must be linearly independent. Hence

$$d_1 - c_1 = \cdots = d_n - c_n = 0. \quad \textbf{QED}$$

§4.4 Coordinate Systems (II)

Suppose $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ is a basis for V and $\mathbf{x}$ is in V. The **coordinates of x relative to the basis $\mathcal{B}$** (or the **$\mathcal{B}$-coordinates of x**) are the weights $c_1, \ldots, c_n$ such that $\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n$.

Coordinate vector of $\mathbf{x}$ is $$[\mathbf{x}]_{\mathcal{B}} \overset{def}{=} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}.$$

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EXAMPLE: Let basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ for $\mathcal{R}^{2\times 2}$ with $\mathbf{b}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Find $[\mathbf{x}]_{\mathcal{B}}$ for vector $\mathbf{x} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$.

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SLN: Let $\mathbf{x} = c_1\,\mathbf{b}_1 + c_2\,\mathbf{b}_2$,

In matrix form $$\begin{bmatrix} 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

Therefore $$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

§4.4 Coordinate Systems (III)

EXAMPLE: Let basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ with $\mathbf{b}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. For vector $\mathbf{x} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$

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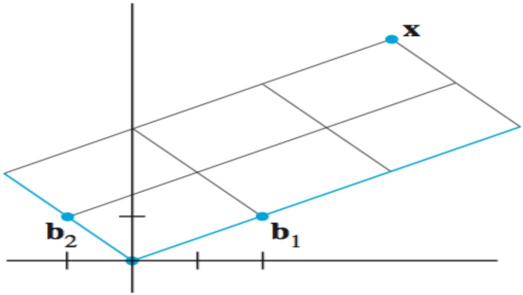

FIGURE 4

The $\mathcal{B}$-coordinate vector of **x** is (3, 2).

§4.4 Coordinate Systems (IV)

Let indexed set $\mathcal{B} = \{\mathbf{b}_1, \cdots, \mathbf{b}_n\}$ be basis for $\mathcal{R}^n$, and let $\mathbf{x} \in \mathcal{R}^n$. The vector equation

$$\mathbf{x} = c_1\,\mathbf{b}_1 + \cdots + c_n\,\mathbf{b}_n$$

is equivalent to

$$\mathbf{x} = P_{\mathcal{B}}\,[\mathbf{x}]_{\mathcal{B}}\,, \quad \text{where} \quad P_{\mathcal{B}} \stackrel{def}{=} \left[\ \mathbf{b}_1, \quad \cdots, \quad \mathbf{b}_n\ \right].$$

$P_{\mathcal{B}}$ is **change-of-coordinates matrix**

§4.4 Coordinate Systems (V)

Let indexed set $\mathcal{B} = \{\mathbf{b}_1, \cdots, \mathbf{b}_n\}$ be basis for vector space V, and let $\mathbf{x} \in V$. The mapping $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$ connects V to $\mathcal{R}^n$.

§4.4 Coordinate Systems (V)

Let indexed set $\mathcal{B} = \{\mathbf{b}_1, \cdots, \mathbf{b}_n\}$ be basis for vector space V, and let $\mathbf{x} \in V$. The mapping $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$ connects V to $\mathcal{R}^n$.

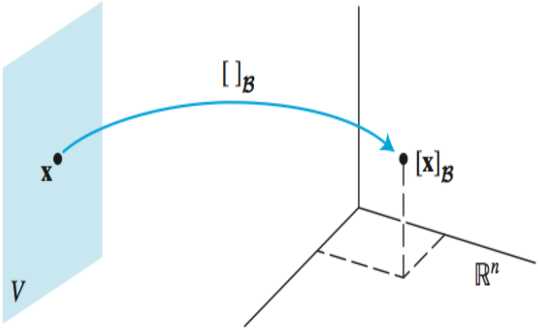

FIGURE 5 The coordinate mapping from V onto $\mathbb{R}^n$.

Example: $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$

Let $V = \mathcal{P}_3$, set of all polynomials of degree at most 3, of form:

$$\mathbf{p}(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3.$$

Let indexed set $\mathcal{B} = \{1, t, t^2, t^3\}$ be basis for $\mathcal{P}_3$. Then

$$[\mathbf{p}]_{\mathcal{B}} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} \in \mathcal{R}^4.$$

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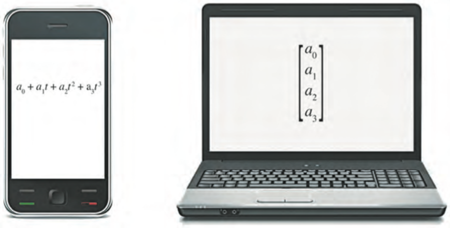

Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a basis for a vector space V. Then the coordinate mapping $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$ is a one-to-one linear transformation from V onto $\mathbb{R}^n$.

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PROOF: Take any two vectors $\mathbf{u}, \mathbf{v} \in V$. Then

$$\mathbf{u} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n, \quad \mathbf{v} = d_1\mathbf{b}_1 + \cdots + d_n\mathbf{b}_n,$$

$$\text{so that} \quad [\mathbf{u}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}, \quad [\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix}.$$

Since $\mathbf{u} + \mathbf{v} = (c_1 + d_1)\mathbf{b}_1 + \cdots + (c_n + d_n)\mathbf{b}_n,$

It follows $[\mathbf{u} + \mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} c_1 + d_1 \\ \vdots \\ c_n + d_n \end{bmatrix} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} + \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix} = [\mathbf{u}]_{\mathcal{B}} + [\mathbf{v}]_{\mathcal{B}}$

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Similarly (book) $[\gamma\,\mathbf{u}]_{\mathcal{B}} = \gamma\,[\mathbf{u}]_{\mathcal{B}}$ for any scalar γ.

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PROOF: Take <u>any</u> two vectors $\mathbf{u}, \mathbf{v} \in V$. Then

$$\mathbf{u} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n, \quad \mathbf{v} = d_1\mathbf{b}_1 + \cdots + d_n\mathbf{b}_n,$$

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Exercises 23/24 for one-to-one proof.