Matrix Transpose

Transpose of matrix A is denoted A^T, and formed by setting each column in A^T from corresponding row in A.

$$\text{Let } A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}.$$

$$\text{Then } A^T = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}, \quad B^T = \begin{bmatrix} 2 & -1 \\ 2 & -1 \\ 1 & 0 \end{bmatrix}.$$

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Theorem: $\left(A^T\right)^T = A, \qquad (A\,B)^T = B^T A^T.$

§2.2 Inverse of Matrix (I)

2×2 system of equations $A\mathbf{x} = \mathbf{b}$: $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}$.

In scalar form:
$$\begin{aligned} a\,x_1 + b\,x_2 &= \beta_1, \quad (\ell_1) \\ c\,x_1 + d\,x_2 &= \beta_2. \quad (\ell_2) \end{aligned}$$

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- $d \times (\ell_1) - b \times (\ell_2) \Longrightarrow (a\,d - c\,b)\,x_1 = d\,\beta_1 - b\,\beta_2.$
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Assume $a\,d - c\,b \neq 0$.

$$\mathbf{x} \;=\; \frac{1}{a\,d - c\,b} \begin{bmatrix} d\,\beta_1 - b\,\beta_2 \\ a\,\beta_2 - c\,\beta_1 \end{bmatrix} = \frac{1}{a\,d - c\,b} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}$$

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2×2 system of equations $A\,\mathbf{x} = \mathbf{b}$: $\left[\begin{array}{cc} a & b \\ c & d \end{array}\right] \left[\begin{array}{c} x_1 \\ x_2 \end{array}\right] = \left[\begin{array}{c} \beta_1 \\ \beta_2 \end{array}\right].$

$$\text{In scalar form:} \quad \begin{array}{rcll} a\,x_1 + b\,x_2 & = & \beta_1, & (\ell_1) \\ c\,x_1 + d\,x_2 & = & \beta_2. & (\ell_2) \end{array}$$

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$$\begin{aligned} \mathbf{x} &= \frac{1}{a\,d - c\,b} \left[\begin{array}{c} d\,\beta_1 - b\,\beta_2 \\ a\,\beta_2 - c\,\beta_1 \end{array}\right] = \frac{1}{a\,d - c\,b} \left[\begin{array}{cc} d & -b \\ -c & a \end{array}\right] \left[\begin{array}{c} \beta_1 \\ \beta_2 \end{array}\right] \\ &= \left(\frac{1}{a\,d - c\,b} \left[\begin{array}{cc} d & -b \\ -c & a \end{array}\right]\right) \mathbf{b} \stackrel{def}{=} A^{-1}\,\mathbf{b}. \end{aligned}$$

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Assume $a\,d - c\,b \neq 0$.

$$\begin{aligned} \mathbf{x} &= \frac{1}{a\,d - c\,b} \begin{bmatrix} d\,\beta_1 - b\,\beta_2 \\ a\,\beta_2 - c\,\beta_1 \end{bmatrix} = \frac{1}{a\,d - c\,b} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} \\ &= \left(\frac{1}{a\,d - c\,b} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \right) \mathbf{b} \stackrel{def}{=} A^{-1}\,\mathbf{b}. \end{aligned}$$

Determinant: $\mathbf{det}\,(A) = a\,d - c\,b$. So A^{-1} exists $\Longleftrightarrow$ $\mathbf{det}\,(A) \neq 0$.

Inverse of Matrix (I)

$$2 \times 2 \text{ matrix} \quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad A^{-1} = \frac{1}{a\,d - c\,b} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Let $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ be the **identity matrix**. Then

$$A\,I = I\,A = A, \quad I\,\mathbf{x} = \mathbf{x} \quad \text{for all } A \text{ and } \mathbf{x}.$$

$$A^{-1}\,A = \left(\frac{1}{a\,d - c\,b} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \right) \begin{bmatrix} a & b \\ c & d \end{bmatrix} = I = A\,A^{-1}.$$

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Definition: Matrix $A \in \mathcal{R}^{n \times n}$ is **invertible** if there exists matrix

$$C \in \mathcal{R}^{n \times n} \text{ so that } \; C\,A = I = A\,C, \text{ with } \; I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \text{ identity.}$$

C is called **inverse** of A, denoted as A^{-1}.

Inverse of Matrix (II)

EX: Let $A = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$, then $A^{-1} = \frac{1}{10}\begin{bmatrix} 4 & -2 \\ -1 & 3 \end{bmatrix}$.

$\begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ has solution $\mathbf{x} = A^{-1}\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 8 \\ -7 \end{bmatrix}$.

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EX: $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}$, then <u>(later show)</u> $A^{-1} = \frac{1}{2}\begin{bmatrix} -9 & 14 & -3 \\ -4 & 8 & -2 \\ 3 & -4 & 1 \end{bmatrix}$.

$A\mathbf{x} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ has solution $\mathbf{x} = A^{-1}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -13 \\ -7 \\ 4 \end{bmatrix}$.

Inverse Matrix (III)

Theorem: Let $A, B \in \mathcal{R}^{n \times n}$ be invertible

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Theorem: Let $A, B \in \mathcal{R}^{n \times n}$ be invertible

- $\left(A^{-1}\right)^{-1} = A$
- $\left(A^{T}\right)^{-1} = \left(A^{-1}\right)^{T}$
- $(A\,B)^{-1} = B^{-1}\,A^{-1}$

PROOF:

$$\begin{aligned}\left(B^{-1}\,A^{-1}\right)(A\,B) &= \left(B^{-1}\right)\,\left(A^{-1}\,A\right)B \\ &= \left(B^{-1}\right)\,B = I\end{aligned}$$

Similarly

$$(A\,B)\,\left(B^{-1}\,A^{-1}\right) = I.$$

Therefore $(A\,B)^{-1} = B^{-1}\,A^{-1}$. **QED**

Elementary Operation $\Longrightarrow$ Elementary Matrix ($E_{1,3}$)

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

- $\textbf{row}_1 \overset{interchange}{\Longleftrightarrow} \textbf{row}_3$

$$A \Longrightarrow \begin{bmatrix} a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\overset{def}{=} \quad E_{1,3}\, A$$

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$$\overset{\textit{def}}{=} \quad E_{1,3}\, A$$

- $E_{1,3}$ obtained by $\mathbf{row}_1 \overset{\textit{interchange}}{\Longleftrightarrow} \mathbf{row}_3$ on I.

Elementary Operation $\Longrightarrow$ Elementary Matrix ($\widehat{E}_2$)

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

- $\mathbf{row}_3 - 2\,\mathbf{row}_1 \Longrightarrow \mathbf{row}_3$

$$A \Longrightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} - 2\,a_{11} & a_{32} - 2\,a_{12} & a_{33} - 2\,a_{13} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \stackrel{def}{=} \widehat{E}_2\,A$$

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$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \overset{def}{=} \widehat{E}_2\, A$$

- $\widehat{E}_2$ obtained by $\mathbf{row}_3 - 2\,\mathbf{row}_1 \Longrightarrow \mathbf{row}_3$ on I.

Elementary Operation $\Longrightarrow$ Elementary Matrix

- Every elementary operation on $A \Longrightarrow E\,A$, where E is result of same EO on identity.

Elementary Operation $\Longrightarrow$ Elementary Matrix

- Every elementary operation on $A \Longrightarrow E\,A$, where E is result of same EO on identity.
- Each elementary matrix E is invertible. E^{-1} is elementary matrix that transforms E to identity.

Reverse Elementary Operation

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

- $\mathbf{row}_1 \overset{interchange}{\Longleftrightarrow} \mathbf{row}_3$

$$A \quad \Longrightarrow \quad E_{1,3}\, A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} A$$

Reverse Elementary Operation

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

- $\mathbf{row}_1 \overset{interchange}{\Longleftrightarrow} \mathbf{row}_3$

$$A \implies E_{1,3}\,A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} A$$

- $E_{1,3}^{-1} = E_{1,3}$.

Inverse of (row interchange) = (same interchange).

Reverse Elementary Operation

- $\mathbf{row}_3 - 2\,\mathbf{row}_1 \Longrightarrow \mathbf{row}_3$

$$A \quad \Longrightarrow \quad \widehat{E}_2\, A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} A$$

Reverse Elementary Operation

- $\mathbf{row}_3 - 2\,\mathbf{row}_1 \Longrightarrow \mathbf{row}_3$

$$A \quad \Longrightarrow \quad \widehat{E}_2\, A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} A$$

- $\widehat{E}_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$

  Inverse of $(\mathbf{row}_3 - 2\,\mathbf{row}_1 \Longrightarrow \mathbf{row}_3)$ is $(\mathbf{row}_3 + 2\,\mathbf{row}_1 \Longrightarrow \mathbf{row}_3)$

Invertible Matrices

Let $A \in \mathcal{R}^{n\times n}$ be invertible (A^{-1} exists). Then in $A\,\mathbf{x} = \mathbf{b}$:

$$\mathbf{x} = I\,\mathbf{x} = \left(A^{-1}\,A\right)\,\mathbf{x} = A^{-1}\,(A\,\mathbf{x}) = A^{-1}\,\mathbf{b}.$$

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- If we know A^{-1}, computing $\mathbf{x}$ is easy.
- Otherwise, can we compute A^{-1}? (YES, but expensive.)

Theorem: $A \in \mathcal{R}^{n\times n}$ Invertible $\iff$ A row reducible to I

PROOF: Let A invertible ($\stackrel{?}{\Longrightarrow}$ A row reducible to I)

- Equation $A\mathbf{x} = \mathbf{b}$ has a solution for EACH $\mathbf{b}$
- A has pivot in every row
- A has no free variables (A square matrix)
- A reducible to I.

Theorem: $A \in \mathcal{R}^{n\times n}$ Invertible $\iff$ A row reducible to I

PROOF: Let A row reducible to I ($\stackrel{?}{\Longrightarrow}$ A invertible)

- Let A be reduced to I by elementary matrices $E_1, \cdots, E_p$

$$E_p\,(E_{p-1}\,(\cdots(E_1\,A))) = I.$$

$$\text{which is} \quad (E_p\,E_{p-1}\cdots E_1)\;A = I.$$

- Therefore

$$A^{-1} = (E_p\,E_{p-1}\cdots E_1) = E_p\,(E_{p-1}\,(\cdots(E_1\,I)))\,.$$

- As you reduce A to I with elementary operations, you turn I to A^{-1}.

Computing A^{-1} vs. Solving $A\mathbf{x} = \mathbf{b}$

- Computing A^{-1}:

$$E_p\left(E_{p-1}\left(\cdots\left(E_1\begin{pmatrix} A & I \end{pmatrix}\right)\right)\right) = \begin{pmatrix} I & A^{-1} \end{pmatrix}.$$

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In practice, only row echelon form needed for $A^{-1}\mathbf{b}$ and A^{-1}

Computing A^{-1} as Solving $AX = I$

- Let $A \in \mathcal{R}^{n \times n}$ be invertible,
- $I = (\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n)$ be the identity matrix,
  where $\mathbf{e}_j$ is 1 at j^{th} component and 0 elsewhere, $1 \le j \le n$.
- $A^{-1} = X = (\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n)$

$$\text{Then} \quad A\,(\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n) = (\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n)\,,$$

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$$\text{or} \quad A\,\mathbf{x}_j = \mathbf{e}_j, \quad j = 1, \cdots n.$$

$$E_p\,(E_{p-1}\,(\cdots(E_1\,(A\;\;(\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n)))))=(I\;\;(A^{-1}\,\mathbf{e}_1, A^{-1}\,\mathbf{e}_2, \cdots, A^{-1}\,\mathbf{e}_n))\,.$$

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More computations to compute A^{-1} than to solve $A\,\mathbf{x} = \mathbf{b}$

§2.3 Characterizations of Invertible Matrices

Let $A \in \mathcal{R}^{n \times n}$ be square matrix. Statements below are equivalent.

a. A is invertible.

d. The equation $A\,\mathbf{x} = \mathbf{0}$ has only trivial solution.

j. There is a matrix $C \in \mathcal{R}^{n \times n}$ so that $C\,A = I$.

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PROOF APPROACH: **a.** $\Longrightarrow$ **j.** $\Longrightarrow$ **d.** $\Longrightarrow$ **a.**

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j. There is a matrix $C \in \mathcal{R}^{n \times n}$ so that $CA = I$.

PROOF OF **a.** $\Longrightarrow$ **j.**:
If A is invertible, then $C = A^{-1}$ works for **j.**

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j. There is a matrix $C \in \mathcal{R}^{n \times n}$ so that $C\,A = I$.

PROOF OF **j.** $\Longrightarrow$ **d.**:
Let $A\,\mathbf{x} = \mathbf{0}$. Then $\mathbf{x}$ must be $\mathbf{0}$ because

$$\mathbf{x} = I\,\mathbf{x} = (C\,A)\,\mathbf{x} = C\,(A\,\mathbf{x}) = C\,\mathbf{0} = \mathbf{0}.$$

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a. A is invertible.

d. The equation $A\mathbf{x} = \mathbf{0}$ has only trivial solution.

j. There is a matrix $C \in \mathcal{R}^{n \times n}$ so that $C\,A = I$.

§2.3 Characterizations of Invertible Matrices

Let $A \in \mathcal{R}^{n\times n}$ be square matrix. Statements below are equivalent.

a. A is invertible.

d. The equation $A\mathbf{x} = \mathbf{0}$ has only trivial solution.

j. There is a matrix $C \in \mathcal{R}^{n\times n}$ so that $C\,A = I$.

PROOF OF **d.** $\Longrightarrow$ **a.**:

Follows from Theorems 11, 12 in §1.9:

- Columns of A must be linearly independent. Therefore
- there must be a pivot in each row. **QED**

§2.3 Inverse Linear Transformation (I)

Let $A \in \mathcal{R}^{n \times n}$ be an invertible matrix. Then

$$A^{-1}(A\mathbf{x}) = \mathbf{x}, \quad A\left(A^{-1}\mathbf{x}\right) = \mathbf{x} \quad \text{for all} \quad \mathbf{x}.$$

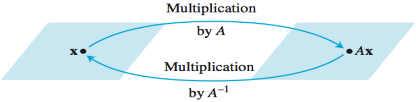

FIGURE 2 A^{-1} transforms $A\mathbf{x}$ back to $\mathbf{x}$.

DEFINITION: A linear transformation $T : \mathcal{R}^n \longrightarrow \mathcal{R}^n$ is **invertible** if there exists function $S : \mathcal{R}^n \longrightarrow \mathcal{R}^n$ so that

$$S(T(\mathbf{x})) = \mathbf{x}, \quad T(S(\mathbf{x})) = \mathbf{x} \quad \text{for all} \quad \mathbf{x} \in \mathcal{R}^n.$$

§2.3 Inverse Linear Transformation (II)

Let A be standard matrix for linear transformation $T: \mathcal{R}^n \longrightarrow \mathcal{R}^n$:

$$T(\mathbf{x}) = A\mathbf{x} \quad \text{for all} \quad \mathbf{x} \in \mathcal{R}^n.$$

Theorem: T is invertible $\iff$ A is invertible.

§3.1 Introduction to Determinants (I)

For 2×2 system of equations $A\mathbf{x} = \mathbf{b}$: $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}$

In scalar form:
$$\begin{aligned} a_{11}\,x_1 + a_{12}\,x_2 &= \beta_1, \quad (\ell_1) \\ a_{21}\,x_1 + a_{22}\,x_2 &= \beta_2. \quad (\ell_2) \end{aligned}$$

§3.1 Introduction to Determinants (I)

For 2×2 system of equations $A\,\mathbf{x} = \mathbf{b}$: $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}$

In scalar form:
$$\begin{array}{rcll} a_{11}\,x_1 + a_{12}\,x_2 & = & \beta_1, & (\ell_1) \\ a_{21}\,x_1 + a_{22}\,x_2 & = & \beta_2. & (\ell_2) \end{array}$$

- $a_{22} \times (\ell_1) - a_{12} \times (\ell_2) \Longrightarrow (a_{11}\,a_{22} - a_{12}\,a_{21})\,x_1 = \square.$
- Similar formula for x_2.

§3.1 Introduction to Determinants (I)

For 2×2 system of equations $A\,\mathbf{x} = \mathbf{b}$: $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}$

In scalar form:
$$\begin{array}{rclr} a_{11}\,x_1 + a_{12}\,x_2 & = & \beta_1, & (\ell_1) \\ a_{21}\,x_1 + a_{22}\,x_2 & = & \beta_2. & (\ell_2) \end{array}$$

- $a_{22} \times (\ell_1) - a_{12} \times (\ell_2) \Longrightarrow (a_{11}\,a_{22} - a_{12}\,a_{21})\,x_1 = \square.$
- Similar formula for x_2.

Determinant of A: **det** $(A) \stackrel{def}{=} a_{11}\,a_{22} - a_{12}\,a_{21} \stackrel{def}{=} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}.$

§3.1 Introduction to Determinants (I)

For 2×2 system of equations $A\,\mathbf{x} = \mathbf{b}$: $\left[\begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \end{array}\right] \left[\begin{array}{c} x_1 \\ x_2 \end{array}\right] = \left[\begin{array}{c} \beta_1 \\ \beta_2 \end{array}\right]$

$$\text{In scalar form:} \quad \begin{array}{rcll} a_{11}\,x_1 + a_{12}\,x_2 & = & \beta_1, & (\ell_1) \\ a_{21}\,x_1 + a_{22}\,x_2 & = & \beta_2. & (\ell_2) \end{array}$$

- $a_{22} \times (\ell_1) - a_{12} \times (\ell_2) \Longrightarrow (a_{11}\,a_{22} - a_{12}\,a_{21})\,x_1 = \square$.
- Similar formula for x_2.

Determinant of A: $\mathbf{det}\,(A) \stackrel{def}{=} a_{11}\,a_{22} - a_{12}\,a_{21} \stackrel{def}{=} \left|\begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \end{array}\right|$.

Solution exists $\Longleftrightarrow$ $\mathbf{det}\,(A) \neq 0$.

Introduction to Determinants (II): 3×3 equations $A\mathbf{x} = \mathbf{b}$

$$
\begin{aligned}
a_{11}\,x_1 + a_{12}\,x_2 + a_{13}\,x_3 &= \beta_1, \quad (\ell_1)\\
a_{21}\,x_1 + a_{22}\,x_2 + a_{23}\,x_3 &= \beta_2. \quad (\ell_2)\\
a_{31}\,x_1 + a_{32}\,x_2 + a_{33}\,x_3 &= \beta_2. \quad (\ell_2)
\end{aligned}
$$

Introduction to Determinants (II): 3×3 equations $A\mathbf{x} = \mathbf{b}$

$$
\begin{aligned}
a_{11}\,x_1 + a_{12}\,x_2 + a_{13}\,x_3 &= \beta_1, & (\ell_1)\\
a_{21}\,x_1 + a_{22}\,x_2 + a_{23}\,x_3 &= \beta_2. & (\ell_2)\\
a_{31}\,x_1 + a_{32}\,x_2 + a_{33}\,x_3 &= \beta_2. & (\ell_2)
\end{aligned}
$$

$$
\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \times (\ell_1) - \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} \times (\ell_2) + \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \times (\ell_3) :
$$

Introduction to Determinants (II): 3×3 equations $A\mathbf{x} = \mathbf{b}$

$$\begin{aligned} a_{11}\,x_1 + a_{12}\,x_2 + a_{13}\,x_3 &= \beta_1, & (\ell_1) \\ a_{21}\,x_1 + a_{22}\,x_2 + a_{23}\,x_3 &= \beta_2. & (\ell_2) \\ a_{31}\,x_1 + a_{32}\,x_2 + a_{33}\,x_3 &= \beta_2. & (\ell_2) \end{aligned}$$

$$\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \times (\ell_1) - \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} \times (\ell_2) + \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \times (\ell_3) :$$

$$\left(a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \right) x_1$$

$$+ \left(a_{12} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{22} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{32} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \right) x_2$$

$$+ \Box\, x_3 = \Box. \qquad (\ell_4)$$

Introduction to Determinants (II): 3×3 equations $A\mathbf{x} = \mathbf{b}$

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= \beta_1, & (\ell_1) \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= \beta_2. & (\ell_2) \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= \beta_2. & (\ell_2) \end{aligned}$$

$$\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \times (\ell_1) - \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} \times (\ell_2) + \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \times (\ell_3) :$$

$$\left(a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \right) x_1$$

$$+ \left(a_{12} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{22} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{32} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \right) x_2$$

$$+ \square\, x_3 = \square. \qquad (\ell_4)$$

Coefficients for x_2, x_3 in $(\ell_4) = 0$. $\mathbf{det}\,(A) \overset{def}{=}$ coefficient for x_1.

Introduction to Determinants (II): 3×3 equations $A\,\mathbf{x} = \mathbf{b}$

$$
\begin{aligned}
a_{11}\,x_1 + a_{12}\,x_2 + a_{13}\,x_3 &= \beta_1, \quad (\ell_1) \\
a_{21}\,x_1 + a_{22}\,x_2 + a_{23}\,x_3 &= \beta_2. \quad (\ell_2) \\
a_{31}\,x_1 + a_{32}\,x_2 + a_{33}\,x_3 &= \beta_2. \quad (\ell_2)
\end{aligned}
$$

$$
\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \times (\ell_1) - \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} \times (\ell_2) + \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \times (\ell_3) :
$$

$$
\left(a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \right) x_1
$$

$$
+ \left(a_{12} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{22} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{32} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \right) x_2
$$

$$
+ \Box\, x_3 = \Box. \qquad (\ell_4)
$$

Coefficients for x_2, x_3 in $(\ell_4) = 0$. $\mathbf{det}\,(A) \overset{def}{=}$ coefficient for x_1.

$$
\boxed{\text{Solution exists} \iff \mathbf{det}\,(A) \neq 0.}
$$

$$
\mathbf{det}\,(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}
$$

$$
= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}
$$

$$\mathbf{det}\,(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

$$= a_{11} \begin{vmatrix} \square & \square & \square \\ \square & a_{22} & a_{23} \\ \square & a_{32} & a_{33} \end{vmatrix} - a_{21} \begin{vmatrix} \square & a_{12} & a_{13} \\ \square & \square & \square \\ \square & a_{32} & a_{33} \end{vmatrix} + a_{31} \begin{vmatrix} \square & a_{12} & a_{13} \\ \square & a_{22} & a_{23} \\ \square & \square & \square \end{vmatrix}$$

Example

$$
\begin{aligned}
\mathbf{det}\,(A) &= \begin{vmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{vmatrix} \\
&= 1 \cdot \begin{vmatrix} 4 & -1 \\ -2 & 0 \end{vmatrix} - 2 \cdot \begin{vmatrix} 5 & 0 \\ -2 & 0 \end{vmatrix} + 0 \cdot \begin{vmatrix} 5 & 0 \\ 4 & -1 \end{vmatrix} \\
&= 1 \cdot (-2) - 2 \cdot (0) + 0 \cdot (-5) \\
&= -2
\end{aligned}
$$

Determinant for $A \in \mathcal{R}^{n\times n}$

$$\mathbf{det}\,(A) = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

$$= a_{11} \begin{vmatrix} \square & \square & \cdots & \square \\ \square & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \square & a_{n2} & \cdots & a_{nn} \end{vmatrix} - a_{21} \begin{vmatrix} \square & a_{12} & \cdots & a_{1n} \\ \square & \square & \cdots & \square \\ \vdots & \vdots & \ddots & \vdots \\ \square & a_{n2} & \cdots & a_{nn} \end{vmatrix} + \cdots$$

$$+ (-1)^{n+1}\, a_{n1} \begin{vmatrix} \square & a_{12} & \cdots & a_{1n} \\ \square & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \square & \square & \cdots & \square \end{vmatrix}$$

Determinant for $A \in \mathcal{R}^{n \times n}$

$$\mathbf{det}(A) = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

$$= a_{11} \begin{vmatrix} \square & \square & \cdots & \square \\ \square & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \square & a_{n2} & \cdots & a_{nn} \end{vmatrix} - a_{21} \begin{vmatrix} \square & a_{12} & \cdots & a_{1n} \\ \square & \square & \cdots & \square \\ \vdots & \vdots & \ddots & \vdots \\ \square & a_{n2} & \cdots & a_{nn} \end{vmatrix} + \cdots$$

$$+ (-1)^{n+1} a_{n1} \begin{vmatrix} \square & a_{12} & \cdots & a_{1n} \\ \square & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \square & \square & \cdots & \square \end{vmatrix}$$

det (A) can be expanded along any row or column.

Example

$$
\begin{aligned}
\mathbf{det}\,(A) &= \begin{vmatrix} 1 & 5 & 7 & 0 \\ 0 & 2 & 4 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 2 & 4 \end{vmatrix} \\
&= 1 \cdot \begin{vmatrix} 2 & 4 & -1 \\ 0 & 1 & 1 \\ 0 & 2 & 4 \end{vmatrix} \\
&= 2 \cdot \begin{vmatrix} 1 & 1 \\ 2 & 4 \end{vmatrix} \\
&= 2 \cdot (2) = 4
\end{aligned}
$$

Upper Triangular Matrix

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix} \in \mathcal{R}^{n\times n}$ be upper triangular.

$$\begin{aligned}
\mathbf{det}\,(A) &= \begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{vmatrix} \\
&= a_{11} \cdot \begin{vmatrix} a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{vmatrix} \\
&= a_{11} \cdot a_{22} \cdots a_{nn}
\end{aligned}$$

§3.2 Properties of Determinants

Let A be a square matrix.

- If a multiple of one row of A is added to another row (ROW REPLACEMENT) to get matrix B, then $\mathbf{det}(B) = \mathbf{det}(A)$.
- If two rows of A are interchanged to (ROW INTERCHANGE) get B, then $\mathbf{det}(B) = -\mathbf{det}(A)$.
- If one row of A is multiplied by k to get B, then $\mathbf{det}(B) = k \cdot \mathbf{det}(A)$.

§3.2 Properties of Determinants

Let A be a square matrix.

- If a multiple of one row of A is added to another row (ROW REPLACEMENT) to get matrix B, then $\mathbf{det}(B) = \mathbf{det}(A)$.
- If two rows of A are interchanged to (ROW INTERCHANGE) get B, then $\mathbf{det}(B) = -\mathbf{det}(A)$.
- If one row of A is multiplied by k to get B, then $\mathbf{det}(B) = k \cdot \mathbf{det}(A)$.

Let A be reduced to echelon form U by row replacements and r row interchanges, then

$$\mathbf{det}(A) = (-1)^r \mathbf{det}(U).$$

Let $A \in \mathcal{R}^{n\times}$ be a square matrix

Thm: If a multiple of one row of A is added to another row (ROW REPLACEMENT) to get matrix B, then $\mathbf{det}\,(B) = \mathbf{det}\,(A)$.

PROOF BY INDUCTION ON n: Proof structure

- Show **Thm** true for $n = 2$
- Assume **Thm** true for $n = k \geq 2$
- Show **Thm** true for $n = k + 1 \geq 3$

Let $A \in \mathcal{R}^{n \times n}$ be square matrix

Thm: If a multiple of one row of A is added to another row to get matrix B, then $\mathbf{det}\left(B\right) = \mathbf{det}\left(A\right)$.

Let $A \in \mathcal{R}^{n \times n}$ be square matrix

Thm: If a multiple of one row of A is added to another row to get matrix B, then $\mathbf{det}\,(B) = \mathbf{det}\,(A)$.

PROOF: For $n = 2$, let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, and $E = \begin{bmatrix} 1 & 0 \\ \lambda & 1 \end{bmatrix}$.

$$B = E\,A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} + \lambda \cdot a_{11} & a_{22} + \lambda \cdot a_{12} \end{bmatrix}$$

is obtained by adding $\lambda \cdot \mathbf{row}_1$ to $\mathbf{row}_2$.

$$\begin{aligned} \mathbf{det}\,(B) &= a_{11} \cdot (a_{22} + \lambda \cdot a_{12}) - a_{12} \cdot (a_{21} + \lambda \cdot a_{11}) \\ &= a_{11} \cdot a_{22} - a_{12} \cdot a_{21} = \mathbf{det}\,(A) = \mathbf{det}\,(E) \cdot \mathbf{det}\,(A) \end{aligned}$$

since $\mathbf{det}\,(E) = 1$.

Let $A \in \mathcal{R}^{n \times n}$ be square matrix

Thm: If a multiple of one row of A is added to another row to get matrix B, then $\mathbf{det}(B) = \mathbf{det}(A)$.

PROOF: For $n = 2$, let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, and $E = \begin{bmatrix} 1 & 0 \\ \lambda & 1 \end{bmatrix}$.

$$B = E\,A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} + \lambda \cdot a_{11} & a_{22} + \lambda \cdot a_{12} \end{bmatrix}$$

is obtained by adding $\lambda \cdot \mathbf{row}_1$ to $\mathbf{row}_2$.

$$\begin{aligned} \mathbf{det}(B) &= a_{11} \cdot (a_{22} + \lambda \cdot a_{12}) - a_{12} \cdot (a_{21} + \lambda \cdot a_{11}) \\ &= a_{11} \cdot a_{22} - a_{12} \cdot a_{21} = \mathbf{det}(A) = \mathbf{det}(E) \cdot \mathbf{det}(A) \end{aligned}$$

since $\mathbf{det}(E) = 1$.

Let $\widehat{E} = \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix}$. Then $\widehat{E}\,A$ is obtained by adding $\lambda \cdot \mathbf{row}_2$ to $\mathbf{row}_1$, and (exercise)

$$\mathbf{det}\left(\widehat{E}\,A\right) = \mathbf{det}\left(\widehat{E}\right) \cdot \mathbf{det}(A) = \mathbf{det}(A).$$

Let $A \in \mathcal{R}^{n\times}$ be a square matrix

Thm: If a multiple of one row of A is added to another row (ROW REPLACEMENT) to get matrix B, then $\mathbf{det}\,(B) = \mathbf{det}\,(A)$.

PROOF: Assume **Thm** true for $n = k \geq 2$

(Sketchy) Proof for $A \in \mathcal{R}^{n \times n}$ with $n = k + 1 \geq 3$

Without loss of generality, assume $\lambda \cdot \textbf{row}_i$ is added to $\textbf{row}_j$ to get $B = E\,A$, with E being identity plus λ in position (j, i), $\boxed{i, j > 1.}$

(Sketchy) Proof for $A \in \mathcal{R}^{n \times n}$ with $n = k + 1 \geq 3$

Without loss of generality, assume $\lambda \cdot \mathbf{row}_i$ is added to $\mathbf{row}_j$ to get $B = E\,A$, with E being identity plus λ in position (j, i), $\boxed{i, j > 1.}$

$$\mathbf{det}\,(A) = a_{11} \begin{vmatrix} \square & \square & \cdots & \square \\ \square & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \square & a_{n2} & \cdots & a_{nn} \end{vmatrix} - a_{12} \begin{vmatrix} \square & \square & \cdots & \square \\ a_{21} & \square & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \square & \cdots & a_{nn} \end{vmatrix} + \cdots + (-1)^{n+1}\, a_{1n} \begin{vmatrix} \square & \square & \cdots & \square \\ a_{21} & a_{22} & \cdots & \square \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & \square \end{vmatrix}$$

(Sketchy) Proof for $A \in \mathcal{R}^{n\times n}$ with $n = k + 1 \geq 3$

Without loss of generality, assume $\lambda \cdot \mathbf{row}_i$ is added to $\mathbf{row}_j$ to get $B = E\,A$, with E being identity plus λ in position (j, i), $\boxed{i, j > 1.}$

$$\mathbf{det}\,(A) = a_{11}\begin{vmatrix} \square & \square & \cdots & \square \\ \square & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \square & a_{n2} & \cdots & a_{nn} \end{vmatrix} - a_{12}\begin{vmatrix} \square & \square & \cdots & \square \\ a_{21} & \square & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \square & \cdots & a_{nn} \end{vmatrix} + \cdots + (-1)^{n+1}\,a_{1n}\begin{vmatrix} \square & \square & \cdots & \square \\ a_{21} & a_{22} & \cdots & \square \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & \square \end{vmatrix}$$

(Induction works on all $k \times k$ determinants; A, B same on row 1)

(Sketchy) Proof for $A \in \mathcal{R}^{n\times n}$ with $n = k+1 \geq 3$

Without loss of generality, assume $\lambda \cdot \mathbf{row}_i$ is added to $\mathbf{row}_j$ to get $B = E\,A$, with E being identity plus λ in position (j,i), $\boxed{i,j>1.}$

$$\mathbf{det}\,(A) = a_{11}\begin{vmatrix} \square & \square & \cdots & \square \\ \square & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \square & a_{n2} & \cdots & a_{nn} \end{vmatrix} - a_{12}\begin{vmatrix} \square & \square & \cdots & \square \\ a_{21} & \square & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \square & \cdots & a_{nn} \end{vmatrix} + \cdots + (-1)^{n+1}\, a_{1n}\begin{vmatrix} \square & \square & \cdots & \square \\ a_{21} & a_{22} & \cdots & \square \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & \square \end{vmatrix}$$

(Induction works on all $k \times k$ determinants; A, B same on row 1)

$$= b_{11}\begin{vmatrix} \square & \square & \cdots & \square \\ \square & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \square & b_{n2} & \cdots & b_{nn} \end{vmatrix} - b_{12}\begin{vmatrix} \square & \square & \cdots & \square \\ b_{21} & \square & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & \square & \cdots & b_{nn} \end{vmatrix} + \cdots + (-1)^{n+1}\, b_{1n}\begin{vmatrix} \square & \square & \cdots & \square \\ b_{21} & b_{22} & \cdots & \square \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & \square \end{vmatrix}$$

$$= \mathbf{det}\,(B)$$

(Sketchy) Proof for $A \in \mathcal{R}^{n \times n}$ with $n = k + 1 \geq 3$

Without loss of generality, assume $\lambda \cdot \mathbf{row}_i$ is added to $\mathbf{row}_j$ to get $B = E\,A$, with E being identity plus λ in position (j, i), $\boxed{i, j > 1.}$

$$\mathbf{det}\,(A) = a_{11} \begin{vmatrix} \square & \square & \cdots & \square \\ \square & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \square & a_{n2} & \cdots & a_{nn} \end{vmatrix} - a_{12} \begin{vmatrix} \square & \square & \cdots & \square \\ a_{21} & \square & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \square & \cdots & a_{nn} \end{vmatrix} + \cdots + (-1)^{n+1}\, a_{1n} \begin{vmatrix} \square & \square & \cdots & \square \\ a_{21} & a_{22} & \cdots & \square \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & \square \end{vmatrix}$$

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$$= \quad \mathbf{det}\,(B)$$

E has determinant 1. So

$$\mathbf{det}\,(E\,A) = \mathbf{det}\,(E)\;\mathbf{det}\,(A)\,.$$

Let A be a square matrix. Let A be reduced to echelon form U by row replacements and r row interchanges, then

$$\mathbf{det}\,(A) = (-1)^r\,\mathbf{det}\,(U)\,.$$

- If A is invertible, then U is upper triangular.

$$\mathbf{det}\,(A) = (-1)^r\,(\text{product of diagonal entries in } U)\,.$$

- If A is not invertible, then U has free variable columns

$$\mathbf{det}\,(A) = 0.$$

$$U = \begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & \blacksquare \end{bmatrix}$$

$\det U \neq 0$

$$U = \begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & 0 & \blacksquare \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

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Let A be a square matrix. Let A be reduced to echelon form U by row replacements and r row interchanges, then

$$\det(A) = (-1)^r \det(U).$$

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$\det U \neq 0$

$$U = \begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & 0 & \blacksquare \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\det U = 0$

A is invertible $\iff \det(A) \neq 0$

Example

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} \quad \text{with one row interchange.}$$

$$\text{So} \quad \mathbf{det}\,(A) = -1 \cdot 1 \cdot 2 = -2.$$

Multiplicative Property

Thm: Let $A, B \in \mathcal{R}^{n\times}$. Then

$$\mathbf{det}\,(A\,B) = \mathbf{det}\,(A)\;\mathbf{det}\,(B)\,.$$

PROOF: If A is not invertible, then neither is $A\,B$ (see Book.)

$$\text{so} \quad \mathbf{det}\,(A\,B) = \mathbf{det}\,(A)\;\mathbf{det}\,(B) = 0.$$

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Assume A is invertible, then A is reducible to I, hence equals product of elementary matrices $E_1, \cdots, E_p$:

$$A \;=\; E_p \cdot E_{p-1} \cdots E_2 \cdot E_1.$$

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$$\begin{aligned} A &= E_p \cdot E_{p-1} \cdots E_2 \cdot E_1. \\ \text{Therefore}\quad A\,B &= E_p \cdot \left(E_{p-1}\left(\cdots\left(E_2 \cdot \left(E_1\,B\right)\right)\right)\right). \end{aligned}$$

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$$\begin{aligned} \mathbf{det}\,(A\,B) &= \mathbf{det}\,(E_p \cdot (E_{p-1}\,(\cdots (E_2 \cdot (E_1\,B))))) \\ &= \mathbf{det}\,(E_p) \cdot \mathbf{det}\,(E_{p-1}\,(\cdots (E_2 \cdot (E_1\,B)))) = \cdots \\ &= \mathbf{det}\,(E_p) \cdot \mathbf{det}\,(E_{p-1}) \cdots \mathbf{det}\,(E_1) \cdot \mathbf{det}\,(B) \\ &= \mathbf{det}\,(E_p \cdots E_1) \cdot \mathbf{det}\,(B) \\ &= \mathbf{det}\,(A)\,\mathbf{det}\,(B) \end{aligned}$$

§3.3 Cramer's Rule: solving $A\mathbf{x} = \mathbf{b}$ for $A \in \mathcal{R}^{n\times n}$

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NOTATION: $\quad A_i(\mathbf{b}) \quad = \quad [\mathbf{a}_1, \cdots, \mathbf{a}_{i-1}, \mathbf{b}, \mathbf{a}_{i+1}, \cdots, \mathbf{a}_n,]\,.$

Thm: Assume A^{-1} exists. Then $x_i = \dfrac{\det(A_i(\mathbf{b}))}{\det(A)}, \quad i = 1, \cdots, n.$

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PROOF: LET $I_i(\mathbf{x}) = [\mathbf{e}_1, \cdots, \mathbf{e}_{i-1}, \mathbf{x}, \mathbf{e}_{i+1}, \cdots, \mathbf{e}_n,]$

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$$I_i(\mathbf{x}) = \begin{bmatrix} 1 & & & x_1 & & & \\ & \ddots & & \vdots & & & \\ & & 1 & x_{i-1} & & & \\ & & & x_i & & & \\ & & & x_{i+1} & 1 & & \\ & & & \vdots & & \ddots & \\ & & & x_n & & & 1 \end{bmatrix}.$$

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$$A\,I_i(\mathbf{x}) = [A\,\mathbf{e}_1, \cdots, A\,\mathbf{e}_{i-1}, A\,\mathbf{x}, A\,\mathbf{e}_{i+1}, \cdots, A\,\mathbf{e}_n,] = A_i(\mathbf{b})\,.$$

§3.3 Cramer's Rule: solving $A\mathbf{x} = \mathbf{b}$ for $A \in \mathcal{R}^{n \times n}$

$$A_i(\mathbf{b}) = [\mathbf{a}_1, \cdots, \mathbf{a}_{i-1}, \mathbf{b}, \mathbf{a}_{i+1}, \cdots, \mathbf{a}_n,].$$

$$\mathbf{det}(I_i(\mathbf{x})) = x_i, \qquad A\, I_i(\mathbf{x}) = A_i(\mathbf{b}).$$

Therefore,

$$\mathbf{det}(A_i(\mathbf{b})) = \mathbf{det}(A\, I_i(\mathbf{x})) = \mathbf{det}(A)\ \mathbf{det}(I_i(\mathbf{x})) = \mathbf{det}(A)\ x_i \qquad .$$

$$\boxed{\text{So} \qquad x_i = \frac{\mathbf{det}(A_i(\mathbf{b}))}{\mathbf{det}(A)}, \quad i = 1, \cdots, n.}$$

§3.3 Cramer's Rule: solving $A\mathbf{x} = \mathbf{b}$ for $A \in \mathcal{R}^{n \times n}$

$$A_i(\mathbf{b}) \quad = \quad [\mathbf{a}_1, \cdots, \mathbf{a}_{i-1}, \mathbf{b}, \mathbf{a}_{i+1}, \cdots, \mathbf{a}_n,] \, .$$

$$\mathbf{det}\left(I_i(\mathbf{x})\right) = x_i, \qquad A\, I_i(\mathbf{x}) = A_i(\mathbf{b}) \, .$$

Therefore,

$$\mathbf{det}\left(A_i(\mathbf{b})\right) = \mathbf{det}\left(A\, I_i(\mathbf{x})\right) = \mathbf{det}\,(A)\ \mathbf{det}\left(I_i(\mathbf{x})\right) = \mathbf{det}\,(A)\ x_i \qquad .$$

$$\boxed{\text{So} \qquad x_i = \frac{\mathbf{det}\left(A_i(\mathbf{b})\right)}{\mathbf{det}\,(A)}, \quad i = 1, \cdots, n.}$$

"For every complex human problem, there is a solution that is neat, simple and wrong" — H. L. Mencken

Cramer's Rule, Example

$$A_i(\mathbf{b}) = [\mathbf{a}_1, \cdots, \mathbf{a}_{i-1}, \mathbf{b}, \mathbf{a}_{i+1}, \cdots, \mathbf{a}_n,].$$

$$\boxed{x_i = \frac{\mathbf{det}(A_i(\mathbf{b}))}{\mathbf{det}(A)}, \quad i = 1, \cdots, n.}$$

Let $A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$. Then $\mathbf{det}(A) = \begin{vmatrix} 1 & 2 \\ -1 & 3 \end{vmatrix} = 5.$

$$\mathbf{det}(A_1(\mathbf{b})) = \begin{vmatrix} -2 & 2 \\ 5 & 3 \end{vmatrix} = -16, \quad \mathbf{det}(A_2(\mathbf{b})) = \begin{vmatrix} 1 & -2 \\ -1 & 5 \end{vmatrix} = 3.$$

So

$$x_1 = \frac{\mathbf{det}(A_1(\mathbf{b}))}{\mathbf{det}(A)} = -\frac{16}{5}, \quad x_2 = \frac{\mathbf{det}(A_2(\mathbf{b}))}{\mathbf{det}(A)} = \frac{3}{5}.$$

§3.3 Determinant as Area

Thm: For $A \in \mathcal{R}^{2\times 2}$, the area of the parallelogram determined by the columns of A is $|\mathbf{det}\,(A)|$.

PROOF:

- If A is not invertible, then $|\mathbf{det}\,(A)| = 0$. Columns of A are parallel, hence parallelogram becomes a line segment, with area $= 0$.

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PROOF:

- If A is not invertible, then $|\mathbf{det}\,(A)| = 0$. Columns of A are parallel, hence parallelogram becomes a line segment, with area $= 0$.
- We now assume If A is invertible in the rest of the proof.

§3.3 Determinant as Area

Thm: For $A \in \mathcal{R}^{2\times 2}$, the area of the parallelogram determined by the columns of A is $|\mathbf{det}\,(A)|$.

PROOF: If $A = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$ is diagonal. Then $|\mathbf{det}\,(A)| = |a\,d|$.

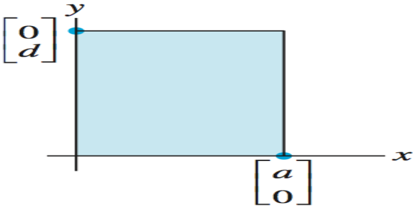

FIGURE 1
Area $= |a\,d|$.

Area of parallelogram is also $|a\,d|$.

§3.3 Determinant as Area

Thm: For $A \in \mathcal{R}^{2\times 2}$, the volume of the parallelogram determined by the columns of A is $|\mathbf{det}\,(A)|$.

PROOF: Let $A \in \mathcal{R}^{2\times 2}$ be invertible. A can be reduced to diagonal matrix with two types of operations:

- interchange two columns.
  This operation does not change $|\mathbf{det}\,(A)|$ or area of the parallelogram.
- **one row** $+\ c \times$ **another** $\Longrightarrow$ **same row**
  This operation does not change $|\mathbf{det}\,(A)|$.
  - Now only need to prove this operation does not change area either.

Let $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 \end{bmatrix}$, and $B = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 + c\,\mathbf{a}_1 \end{bmatrix}$. Then $|\mathbf{det}\,(A)| = |\mathbf{det}\,(B)|$

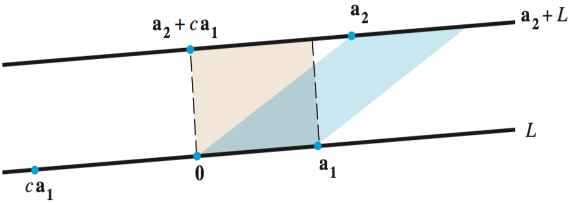

FIGURE 2 Two parallelograms of equal area.

- L is a line through $\mathbf{0}$ and $\mathbf{a}_1$.
- $\mathbf{a}_2 + L$ is a line through $\mathbf{a}_2$ and parallel to L.
- Both parallelograms have same base and height, hence same area.

§3.3 Determinant as Volume

Thm: For $A \in \mathcal{R}^{3\times 3}$, the volume of the parallelepiped determined by the columns of A is $|\mathbf{det}\,(A)|$.

PROOF: If $A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$ is diagonal. Then

$|\mathbf{det}\,(A)| = |a\,b\,c|$.

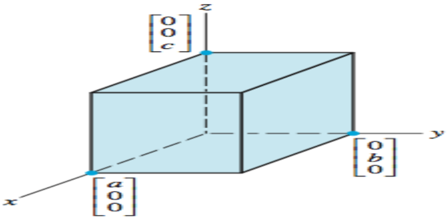

FIGURE 3
Volume $= |abc|$.

Volume of parallelepiped is also $|a\,b\,c|$.

§3.3 Determinant as Volume

Thm: For $A \in \mathcal{R}^{3\times3}$, the volume of the parallelepiped determined by the columns of A is $|\mathbf{det}\,(A)|$.

PROOF: Let $A \in \mathcal{R}^{3\times3}$ be invertible. A can be reduced to diagonal matrix with two types of operations:

- interchange two columns.
  This operation does not change $|\mathbf{det}\,(A)|$ or area of the parallelogram.
- **one row** $+ \, c \times$ **another** $\Longrightarrow$ **same row**
  This operation does not change $|\mathbf{det}\,(A)|$.
  - Now only need to prove this operation does not change area either.

Let $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{bmatrix}$, and $B = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 + c\,\mathbf{a}_1 & \mathbf{a}_3 \end{bmatrix}$. Then $|\mathbf{det}\,(A)| = |\mathbf{det}\,(B)|$

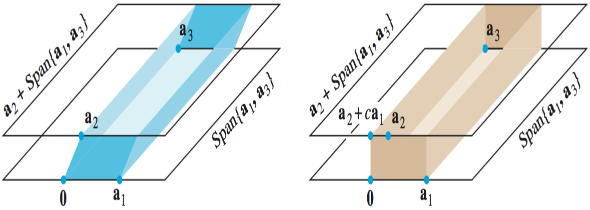

FIGURE 4 Two parallelepipeds of equal volume.

- Base in $\mathbf{Span}\,(\mathbf{a}_1, \mathbf{a}_3)$.
- $\mathbf{a}_2 + \mathbf{Span}\,(\mathbf{a}_1, \mathbf{a}_3)$ is a plane parallel $\mathbf{Span}\,(\mathbf{a}_1, \mathbf{a}_3)$.
- Both parallelepipeds have same base and height, hence same volume.

Vector Space and Subspace

- **Vector Space** is a set V of objects (vectors)
- OPERATIONS: *addition* and *scalar multiplication*
- Axioms below work for all $\mathbf{u}, \mathbf{v} \in V$ and all scalars.

1. The sum of $\mathbf{u}$ and $\mathbf{v}$, denoted by $\mathbf{u} + \mathbf{v}$, is in V.
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
3. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$.
4. There is a **zero** vector $\mathbf{0}$ in V such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$.
5. For each $\mathbf{u}$ in V, there is a vector $-\mathbf{u}$ in V such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.
6. The scalar multiple of $\mathbf{u}$ by c, denoted by $c\mathbf{u}$, is in V.
7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$.
8. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$.
9. $c(d\mathbf{u}) = (cd)\mathbf{u}$.
10. $1\mathbf{u} = \mathbf{u}$.

Vector Space: Example (I)

1. The sum of $\mathbf{u}$ and $\mathbf{v}$, denoted by $\mathbf{u}+\mathbf{v}$, is in V.
2. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$.
3. $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$.
4. There is a **zero** vector $\mathbf{0}$ in V such that $\mathbf{u}+\mathbf{0}=\mathbf{u}$.
5. For each $\mathbf{u}$ in V, there is a vector $-\mathbf{u}$ in V such that $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$.
6. The scalar multiple of $\mathbf{u}$ by c, denoted by $c\mathbf{u}$, is in V.
7. $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$.
8. $(c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u}$.
9. $c(d\mathbf{u})=(cd)\mathbf{u}$.
10. $1\mathbf{u}=\mathbf{u}$.

$V=\mathcal{R}^n$.

Vector Space: Example (II)

1. The sum of $\mathbf{u}$ and $\mathbf{v}$, denoted by $\mathbf{u}+\mathbf{v}$, is in V.
2. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$.
3. $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$.
4. There is a **zero** vector $\mathbf{0}$ in V such that $\mathbf{u}+\mathbf{0}=\mathbf{u}$.
5. For each $\mathbf{u}$ in V, there is a vector $-\mathbf{u}$ in V such that $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$.
6. The scalar multiple of $\mathbf{u}$ by c, denoted by $c\mathbf{u}$, is in V.
7. $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$.
8. $(c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u}$.
9. $c(d\mathbf{u})=(cd)\mathbf{u}$.
10. $1\mathbf{u}=\mathbf{u}$.

$V=\mathcal{P}_3$, set of all polynomials of degree at most 3, of form:

$$\mathbf{p}(t)=a_0+a_1 t+a_2 t^2+a_3 t^3.$$

Vector Space: Example (II)

1. The sum of **u** and **v**, denoted by $\mathbf{u} + \mathbf{v}$, is in V.
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
3. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$.
4. There is a **zero** vector $\mathbf{0}$ in V such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$.
5. For each **u** in V, there is a vector $-\mathbf{u}$ in V such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.
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9. $c(d\mathbf{u}) = (cd)\mathbf{u}$.
10. $1\mathbf{u} = \mathbf{u}$.

$V = \mathcal{P}_3$, set of all polynomials of degree at most 3, of form:

$$\mathbf{p}(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3.$$

- ADDITION: Let $\mathbf{q}(t) = b_0 + b_1 t + b_2 t^2 + b_3 t^3$.

$$(\mathbf{p} + \mathbf{q})(t) \overset{def}{=} (a_0 + b_0) + (a_1 + b_1)\, t + (a_2 + b_2)\, t^2 + (a_3 + b_3)\, t^3 \in \mathcal{P}_3$$

- SCALAR MULTIPLICATION:

$$(\alpha\, \mathbf{p})(t) \overset{def}{=} (\alpha\, a_0) + (\alpha\, a_1)\, t + (\alpha\, a_2)\, t^2 + (\alpha\, a_3)\, t^3 \in \mathcal{P}_3$$

Vector Space is a set

If needles were vectors, then the cactus would be vector space.

Vectors can point to all possible directions, have all possible sizes.