§1.4 Matrix Equation $A\mathbf{x} = \mathbf{b}$: Linear Combination (I)

If A is an $m \times n$ matrix, with columns $\mathbf{a}_1, \dots, \mathbf{a}_n$, and if $\mathbf{x}$ is in $\mathbb{R}^n$, then the **product of A and $\mathbf{x}$**, denoted by $A\mathbf{x}$, is **the linear combination of the columns of A using the corresponding entries in $\mathbf{x}$ as weights**; that is,

$$A\mathbf{x} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n$$

Matrix-vector Product $\Longleftrightarrow$ Linear Combination (II)

Example:

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix}\begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix} = 4\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 2 \\ -5 \end{bmatrix} + 7\begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ 0 \end{bmatrix} + \begin{bmatrix} 6 \\ -15 \end{bmatrix} + \begin{bmatrix} -7 \\ 21 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

Linear Equations in terms of Matrix-vector Product

Example

$$
\begin{aligned}
x_1 + 2x_2 - x_3 &= 4 \\
-5x_2 + 3x_3 &= 1
\end{aligned}
$$

equivalent to

$$
x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ -5 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}
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Linear Equations in terms of Matrix-vector Product

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$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

Linear Equations in terms of Linear Combinations

If A is an $m \times n$ matrix, with columns $\mathbf{a}_1, \ldots, \mathbf{a}_n$, and if $\mathbf{b}$ is in $\mathbb{R}^m$, the matrix equation

$$A\mathbf{x} = \mathbf{b} \tag{4}$$

has the same solution set as the vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b} \tag{5}$$

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The equation $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $\mathbf{b}$ is a linear combination of the columns of A.

Existence of Solutions

Example: Let $A = \begin{bmatrix} 1 & 3 & 4 \\ -4 & 2 & -6 \\ -3 & -2 & -7 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$.

QUESTION: For what values of b_1, b_2, b_3 is equation $A\mathbf{x} = \mathbf{b}$ consistent?

$$\left[\begin{array}{ccc|c} 1 & 3 & 4 & b_1 \\ -4 & 2 & -6 & b_2 \\ -3 & -2 & -7 & b_3 \end{array}\right] \overset{\substack{(\ell_2)+(4)\times(\ell_1)\to(\ell_2) \\ (\ell_3)+(3)\times(\ell_1)\to(\ell_3)}}{\Longrightarrow}$$

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ANSWER: equation $A\mathbf{x} = \mathbf{b}$ consistent $\Longleftrightarrow$ $b_3 - 1/2\,b_2 + b_1 = 0.$

Equivalent Statements on Existence of Solution (I)

a. For each $\mathbf{b}$ in $\mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution.

b. Each $\mathbf{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of A.

c. The columns of A span $\mathbb{R}^m$.

d. A has a pivot position in every row.

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Will show $(a) \iff (d)$

Equivalent Statements (II): $(d) \iff (a)$

- A has pivot in each row $\stackrel{?}{\Longrightarrow}$ $A\mathbf{x} = \mathbf{b}$ has solution for each $\mathbf{b}$:

$$\left(A \quad \middle| \; \mathbf{b}\right) \quad \stackrel{\text{row echelon}}{\Longrightarrow} \quad \left(U \quad \middle| \; \mathbf{d}\right),$$

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$$\Longrightarrow \text{Let } \mathbf{d} \text{ be vector with all components } 1.$$

(work row echelon backwards, CAN find corresponding $\mathbf{b}$)

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$\Longrightarrow$ A has a pivot in each row. YES!

§1.5 Solution Sets of Linear Systems:

Homogeneous Systems $A\mathbf{x} = \mathbf{0}$

trivial solution: $\mathbf{x} = \mathbf{0}$; any non-zero solution $\mathbf{x}$ is **non-trivial**.

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Example:

$$\begin{aligned} 3x_1 + 5x_2 - 4x_3 &= 0, \\ -3x_1 - 2x_2 + 4x_3 &= 0, \\ 6x_1 + x_2 - 8x_3 &= 0. \end{aligned}$$

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Augmented matrix $(A \mid \mathbf{b})$ to row echelon form

$$\begin{pmatrix} 3 & 5 & -4 & 0 \\ -3 & -2 & 4 & 0 \\ 6 & 1 & -8 & 0 \end{pmatrix} \sim \begin{pmatrix} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & -9 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

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x_3 is free variable.

Homogeneous Systems Example (I)

Augmented matrix in row echelon form

$$\begin{pmatrix} 3 & 5 & -4 & & 0 \\ 0 & 3 & 0 & & 0 \\ 0 & 0 & 0 & & 0 \end{pmatrix}$$

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Solution is $$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} \frac{4}{3} \\ 0 \\ 1 \end{pmatrix},$$

Homogeneous Systems Example (I)

Augmented matrix in row echelon form

$$\begin{pmatrix} 3 & 5 & -4 & & 0 \\ 0 & 3 & 0 & & 0 \\ 0 & 0 & 0 & & 0 \end{pmatrix}$$

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$$\text{Solution is} \qquad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} \frac{4}{3} \\ 0 \\ 1 \end{pmatrix},$$

$$\text{Set of Solutions} \qquad = \textbf{span}\left(\begin{pmatrix} \frac{4}{3} \\ 0 \\ 1 \end{pmatrix} \right).$$

Homogeneous Systems Example (II)

Example 2: Augmented matrix in row echelon form

$$\begin{pmatrix} 1 & 1 & 1 & 1 & & 0 \\ 0 & 0 & 2 & 4 & & 0 \end{pmatrix}$$

x_2, x_4 are free variables.

$$\text{Solution is} \qquad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = x_2 \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 1 \\ 0 \\ -2 \\ 1 \end{pmatrix},$$

Homogeneous Systems Example (II)

Example 2: Augmented matrix in row echelon form

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$$\text{Set of Solutions} \quad = \textbf{span}\left(\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -2 \\ 1 \end{pmatrix} \right).$$

In general, set of solutions to $A\mathbf{x} = \mathbf{0}$ is $\textbf{span}\,(\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_p)$, where p is number of free variables.

§1.5 Solution Sets of Linear Systems:

NONHOMOGENEOUS SYSTEMS $A\mathbf{x} = \mathbf{b} \quad (\neq \mathbf{0})$

Example 3: Augmented matrix for $A\mathbf{x} = \mathbf{b}$:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & & 1 \\ 0 & 0 & 2 & 4 & & -2 \end{pmatrix}$$

x_2, x_4 are free variables.

$$\text{Solution is} \quad \mathbf{x} \quad = \quad \underbrace{\begin{pmatrix} 2 \\ 0 \\ -1 \\ 0 \end{pmatrix}} + \underbrace{x_2 \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 1 \\ 0 \\ -2 \\ 1 \end{pmatrix}},$$

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$$\stackrel{def}{=} \quad \mathbf{p} \quad + \quad \mathbf{v}_h,$$

$$\text{where} \qquad A\mathbf{p} = \mathbf{b}, \quad A\mathbf{v}_h = \mathbf{0}.$$

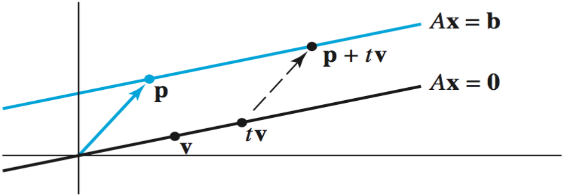

FIGURE 5 Parallel solution sets of $A\mathbf{x} = \mathbf{b}$ and $A\mathbf{x} = \mathbf{0}$.

Suppose the equation $A\mathbf{x} = \mathbf{b}$ is consistent for some given $\mathbf{b}$, and let $\mathbf{p}$ be a solution. Then the solution set of $A\mathbf{x} = \mathbf{b}$ is the set of all vectors of the form $\mathbf{w} = \mathbf{p} + \mathbf{v}_h$, where $\mathbf{v}_h$ is any solution of the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

§1.7 Linear Independence

An indexed set of vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is said to be **linearly independent** if the vector equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_p\mathbf{v}_p = \mathbf{0}$$

has only the trivial solution. The set $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is said to be **linearly dependent** if there exist weights $c_1, \dots, c_p$, not all zero, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p = \mathbf{0} \tag{2}$$

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Since $x_1\,\mathbf{v}_1 + x_2\,\mathbf{v}_2 + \cdots + x_p\,\mathbf{v}_p = \left(\ \mathbf{v}_1,\ \ \mathbf{v}_2,\ \ \cdots,\ \ \mathbf{v}_p\ \right)\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{pmatrix},$

Vectors $\{\ \mathbf{v}_1,\ \ \mathbf{v}_2,\ \ \cdots,\ \ \mathbf{v}_p\ \}$ are linear independent $\Longleftrightarrow$

system of equations $\left(\ \mathbf{v}_1,\ \ \mathbf{v}_2,\ \ \cdots,\ \ \mathbf{v}_p\ \right)\mathbf{x} = \mathbf{0}$

does NOT have non-trivial solution.

Linear Independence: Examples (I)

- Let $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$. Determine if they are Linearly Independent.

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  SOLUTION: Form matrix and do row echelon

$$\left(\begin{array}{ccc|c} \mathbf{v}_1, & \mathbf{v}_2, & \mathbf{v}_3 & \mathbf{0} \end{array} \right) = \left(\begin{array}{ccc|c} 1 & 4 & 2 & 0 \\ 2 & 5 & 1 & 0 \\ 3 & 6 & 0 & 0 \end{array} \right)$$

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  SOLUTION: Form matrix and do row echelon

$$\left(\mathbf{v}_1, \quad \mathbf{v}_2, \quad \mathbf{v}_3 \;\middle|\; \mathbf{0} \right) = \left(\begin{array}{ccc|c} 1 & 4 & 2 & 0 \\ 2 & 5 & 1 & 0 \\ 3 & 6 & 0 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 4 & 2 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Linear Independence: Examples (I)

- Let $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$. Determine if they are Linearly Independent.

  SOLUTION: Form matrix and do row echelon

$$\begin{pmatrix} \mathbf{v}_1, & \mathbf{v}_2, & \mathbf{v}_3 & \Big| & \mathbf{0} \end{pmatrix} = \left(\begin{array}{ccc|c} 1 & 4 & 2 & 0 \\ 2 & 5 & 1 & 0 \\ 3 & 6 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 4 & 2 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

  Free-variable is x_3. Choose $x_3 = 1$, solution is $\mathbf{x} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$,

  so: $\quad 2\,\mathbf{v}_1 - \mathbf{v}_2 + \mathbf{v}_3 = \mathbf{0}$. Linearly Dependent

Linear Independence: Examples (II)

- Determine if columns of matrix $\begin{pmatrix} 0 & 1 & 4 \\ 1 & 2 & -1 \\ 5 & 8 & 0 \end{pmatrix}$ are Linearly Independent.

Linear Independence: Examples (II)

- Determine if columns of matrix $\begin{pmatrix} 0 & 1 & 4 \\ 1 & 2 & -1 \\ 5 & 8 & 0 \end{pmatrix}$ are Linearly Independent.
  SOLUTION: Do row echelon for $A\mathbf{x} = \mathbf{0}$:

$$\left(\begin{array}{ccc|c} 0 & 1 & 4 & 0 \\ 1 & 2 & -1 & 0 \\ 5 & 8 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 13 & 0 \end{array}\right)$$

Linear Independence: Examples (II)

- Determine if columns of matrix $\begin{pmatrix} 0 & 1 & 4 \\ 1 & 2 & -1 \\ 5 & 8 & 0 \end{pmatrix}$ are Linearly Independent.

  SOLUTION: Do row echelon for $A\mathbf{x} = \mathbf{0}$:

$$\left(\begin{array}{ccc|c} 0 & 1 & 4 & 0 \\ 1 & 2 & -1 & 0 \\ 5 & 8 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 13 & 0 \end{array}\right)$$

  No free-variable. Columns of matrix Linearly Independent

Linear Independence: One vector

Let $\mathbf{v}$ be a vector, c scalar, and

$$c\,\mathbf{v} = \mathbf{0}.$$

- If $\mathbf{v} \neq \mathbf{0}$, then c must be 0, $\{\mathbf{v}\}$ Linearly Independent.
- If $\mathbf{v} = \mathbf{0}$, then $c = 1$, $\{\mathbf{v}\}$ Linearly Dependent.

Linear Independence: Two vectors

Let $\mathbf{v}_1, \mathbf{v}_2$ be two vectors, c_1, c_2 scalars, and

$$c_1\,\mathbf{v}_1 + c_2\,\mathbf{v}_2 = \mathbf{0}.$$

If $\{\mathbf{v}_1, \mathbf{v}_2\}$ linearly dependent,

- Either $c_1 \neq 0$, then

$$\mathbf{v}_1 = -\left(c_2/c_1\right)\,\mathbf{v}_2$$

- Or $c_2 \neq 0$, then

$$\mathbf{v}_2 = -\left(c_1/c_2\right)\,\mathbf{v}_1$$

$\boxed{\mathbf{v}_1, \mathbf{v}_2 \text{ linearly dependent} \iff \text{one is multiple of the other.}}$

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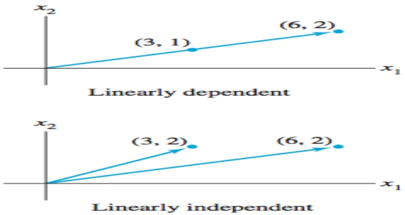

Linear Independence: At least Two vectors (I)

Let $\mathcal{S} = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_p\}$ be a set of $p \geq 2$ vectors.

- $\mathcal{S}$ is linearly dependent $\iff$ one vector in $\mathcal{S}$ is linear combination of others.

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Example: Let $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$.

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Therefore: $\mathbf{v}_3 = -2\,\mathbf{v}_1 + \mathbf{v}_2$, for $j = 3$.

Linear Independence: At least Two vectors (II)

Let $\mathcal{S} = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_p\}$ be a set of $p \geq 2$ vectors.

- **Theorem:** If $\mathcal{S}$ is linearly dependent and $\mathbf{v}_1 \neq \mathbf{0}$, then some $\mathbf{v}_j$ is linear combination of $\mathbf{v}_1, \cdots, \mathbf{v}_{j-1}$.

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Proof: Let

$$c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \cdots + c_p \mathbf{v}_p = \mathbf{0}, \quad (\ell)$$

with some of $c_1, c_2, \cdots, c_p \neq 0$.

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Let j be the largest subscript for which $c_j \neq 0$. If $j = 1$, then (ℓ) becomes

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So $j > 1$, and (ℓ) becomes

$$\mathbf{v}_j = -\,(c_1/c_j)\ \mathbf{v}_1 - (c_2/c_j)\ \mathbf{v}_2 + \cdots + (c_{j-1}/c_j)\,\mathbf{v}_{j-1}. \quad \textbf{QED}$$

Linear Independence (III)

Let $\mathcal{S} = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_p\}$ be a set of vectors.

- **Theorem:** If $\mathcal{S}$ contains the $\mathbf{0}$ vector, then $\mathcal{S}$ is linearly dependent.

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Proof: Let $\mathbf{v}_j = \mathbf{0}$ for some index j, then

$$0 \times \mathbf{v}_1 + \cdots + 0 \times \mathbf{v}_{j-1} + 1 \times \mathbf{v}_j + 0 \times \mathbf{v}_{j+1} + 0 \times \mathbf{v}_p = \mathbf{0}.$$

Linear Independence (IV)

Let $\mathcal{S} = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_p\}$ be a set of p vectors.

- **Theorem:** If vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_p$ contains n components each with $n < p$, then $\mathcal{S}$ is linearly dependent.

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Thus the row echelon form for the augmented matrix $(A \mid \mathbf{0})$ must have free variable columns.
Therefore $A\mathbf{x} = \mathbf{0}$ must have a non-trivial solution, and columns of A linearly dependent. **QED**

Linear Independence (V)

Let $\mathcal{S} = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_p\}$ be a set of p vectors.

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Example: vectors $\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$ are linearly dependent for any α_1, α_2.

Introduction to Linear Transformations: Example

Given $A = \begin{bmatrix} 4 & -3 & 1 & 3 \\ 2 & 0 & 5 & 1 \end{bmatrix}$, linear transform is a function:

$$\boxed{A\left(4\_component\_vector\right) = \left(2\_component\_vector\right).}$$

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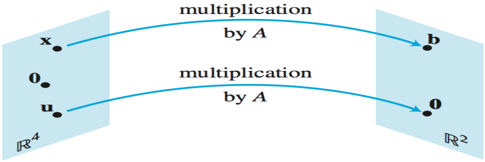

FIGURE 1 Transforming vectors via matrix multiplication.

Linear Transformations

Matrix Transformation: Definition

Given matrix $A \in \mathcal{R}^{m \times n}$, **matrix transform** is function from $\mathcal{R}^n$ to $\mathcal{R}^m$:

$$\text{For each } \mathbf{x} \in \mathcal{R}^n, \quad T(\mathbf{x}) \stackrel{def}{=} A\mathbf{x} \quad (\in \mathcal{R}^m).$$

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Ex: Let $A = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}$. Def matrix transformation $T : \mathcal{R}^2 \longrightarrow \mathcal{R}^3$:

$$T(\mathbf{x}) = A\,\mathbf{x} = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 5 \\ 7 \end{bmatrix} = \underbrace{\begin{bmatrix} x_1 - 3\,x_2 \\ 3\,x_1 + 5\,x_2 \\ -x_1 + 7\,x_2 \end{bmatrix}}_{\textbf{image}}$$

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$$\textbf{range of } T = \textbf{span}\left(\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} -3 \\ 5 \\ 7 \end{bmatrix} \right).$$

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SOLUTION:

$$T(\mathbf{u}) = A\,\mathbf{u} = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ -9 \end{bmatrix}.$$

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- Find $\mathbf{x} \in \mathcal{R}^2$ whose image under T is $\mathbf{b} = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$.
- Is there more than one $\mathbf{x}$ whose image under T is $\mathbf{b}$?

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SOLUTION: $T(\mathbf{x}) = \mathbf{b} \iff A\mathbf{x} = \mathbf{b}$. Row echelon on $(A \mid \mathbf{b})$:

$$\left[\begin{array}{cc|c} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & -5 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -3 & 3 \\ 0 & 1 & -0.5 \\ 0 & 0 & 0 \end{array}\right].$$

pre-image $\mathbf{x} = \begin{bmatrix} 1.5 \\ -0.5 \end{bmatrix}$ is UNIQUE.

Matrix Transformation: Example

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SOLUTION: c is in the range of $T \iff c$ is image of some $\mathbf{x} \in \mathcal{R}^2$. Let $A\mathbf{x} = \mathbf{c}$. Row echelon on $(A \mid \mathbf{c})$:

$$\left[\begin{array}{cc|c} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & 5 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -3 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & -35 \end{array}\right].$$

Equations have no solution, c is NOT in the range of T.

LINEAR Transformation: Definition

Given matrix $A \in \mathcal{R}^{m \times n}$, a **transformation** is a function from

$$\mathcal{R}^n \quad (= \textbf{domain}) \quad \text{to} \quad \mathcal{R}^m \quad (= \textbf{codomain})$$

A transformation (or mapping) T is **linear** if:

(i) $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v}$ in the domain of T;

(ii) $T(c\mathbf{u}) = cT(\mathbf{u})$ for all scalars c and all $\mathbf{u}$ in the domain of T.

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- Matrix Transformation is linear transformation.
- ...

What Transformation is NOT Linear

Linear Transformation: Simple Facts

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- (i) + (ii)

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- Repeat on (ℓ)

$$T(c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_p\mathbf{u}_p) = c_1 T(\mathbf{u}_1) + c_2 T(\mathbf{u}_2) + \cdots + c_p T(\mathbf{u}_p)$$

Example Linear Transformation $T : \mathcal{R}^2 \rightarrow \mathcal{R}^2$

$$T(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}$$

Find the images under T of $\mathbf{u} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, and

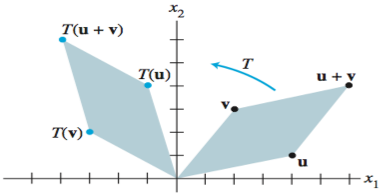
$\mathbf{u} + \mathbf{v} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$.

Example Linear Transformation $T : \mathcal{R}^2 \to \mathcal{R}^2$

$$T(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}$$

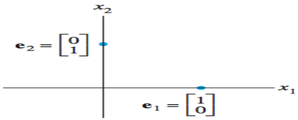
Find the images under T of $\mathbf{u} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, and $\mathbf{u} + \mathbf{v} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$. SOLUTION: T is a flip-reflection

$$T(\mathbf{u}) = \begin{bmatrix} -1 \\ 4 \end{bmatrix}, T(\mathbf{v}) = \begin{bmatrix} -3 \\ 2 \end{bmatrix}, T(\mathbf{u}+\mathbf{v}) = \begin{bmatrix} -4 \\ 6 \end{bmatrix}.$$



§1.9 The Matrix of a Linear Transformation

Motivating **example**: Define $\mathbf{e}_1, \mathbf{e}_2$ below.



Let linear transformation $T : \mathcal{R}^2 \to \mathcal{R}^3$ satisfy

$$T(\mathbf{e}_1) = \begin{bmatrix} 5 \\ -7 \\ 2 \end{bmatrix} \quad \text{and} \quad T(\mathbf{e}_2) = \begin{bmatrix} -3 \\ 8 \\ 2 \end{bmatrix}.$$

Find formula for image of arbitrary $\mathbf{x} \in \mathcal{R}^2$.

$$T(\mathbf{e}_1) = \begin{bmatrix} 5 \\ -7 \\ 2 \end{bmatrix} \quad \text{and} \quad T(\mathbf{e}_2) = \begin{bmatrix} -3 \\ 8 \\ 2 \end{bmatrix}.$$

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$$T(\mathbf{e}_1) = \begin{bmatrix} 5 \\ -7 \\ 2 \end{bmatrix} \quad \text{and} \quad T(\mathbf{e}_2) = \begin{bmatrix} -3 \\ 8 \\ 2 \end{bmatrix}.$$

Find formula for image of arbitrary $\mathbf{x} \in \mathcal{R}^2$.

SOLUTION: Write

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = x_1\,\mathbf{e}_1 + x_2\,\mathbf{e}_2.$$

$$T(\mathbf{e}_1) = \begin{bmatrix} 5 \\ -7 \\ 2 \end{bmatrix} \quad \text{and} \quad T(\mathbf{e}_2) = \begin{bmatrix} -3 \\ 8 \\ 2 \end{bmatrix}.$$

Find formula for image of arbitrary $\mathbf{x} \in \mathcal{R}^2$.

SOLUTION: Write

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = x_1\,\mathbf{e}_1 + x_2\,\mathbf{e}_2.$$

Therefore

$$\begin{aligned} T(\mathbf{x}) &= x_1\,T(\mathbf{e}_1) + x_2\,T(\mathbf{e}_2) = x_1 \begin{bmatrix} 5 \\ -7 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 8 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} 5x_1 - 3x_2 \\ -7x_1 + 8x_2 \\ 2x_1 \end{bmatrix} = \underbrace{\begin{bmatrix} 5 & -3 \\ -7 & 8 \\ 2 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\mathbf{x}} \\ &= \begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) \end{bmatrix} \mathbf{x} \end{aligned}$$

Standard Matrix

- Let $T : \mathcal{R}^n \rightarrow \mathcal{R}^m$ be linear transformation.
- Let $\mathbf{e}_j \in \mathcal{R}^n$ be 1 at j^{th} entry and 0 elsewhere, $1 \leq j \leq n$.

$$\boxed{\text{Then} \quad T(\mathbf{x}) = A\,\mathbf{x}, \quad \text{for all} \;\; \mathbf{x} \in \mathcal{R}^n,}$$

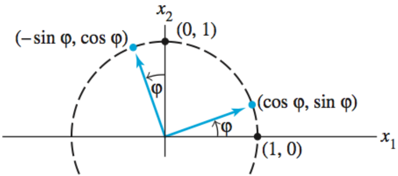
with **Standard Matrix**

$$A \stackrel{def}{=} \left[\begin{array}{cccc} T(\mathbf{e}_1) & T(\mathbf{e}_2) & \cdots & T(\mathbf{e}_n) \end{array}\right] \in \mathcal{R}^{m \times n}.$$

Example: Givens Rotation in $\mathcal{R}^2$

$$T(\mathbf{x}) = A\,\mathbf{x},$$

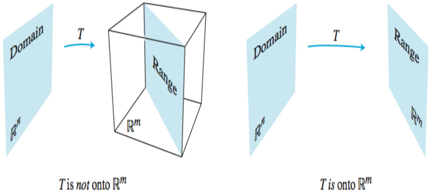
$$\text{with} \quad A = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}$$



T rotates $\mathbf{x}$ counter clock-wise by angle φ.

onto and **one-to-one** (I)

- $T : \mathcal{R}^n \to \mathcal{R}^m$ is **onto** if each $\mathbf{b} \in \mathcal{R}^m$ is image of at least one $\mathbf{x} \in \mathcal{R}^n$.



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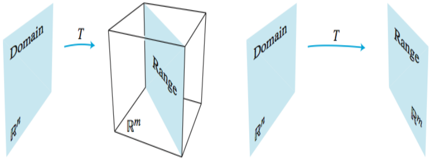
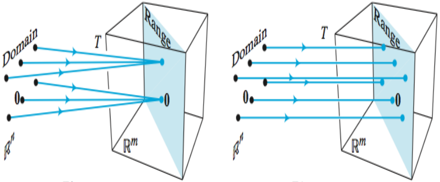


T is *not* onto $\mathbb{R}^m$ *T* is onto $\mathbb{R}^m$

- $T : \mathcal{R}^n \rightarrow \mathcal{R}^m$ is **one-to-one** if each $\mathbf{b} \in \mathcal{R}^m$ is image of at most one $\mathbf{x} \in \mathcal{R}^n$.



T is *not* one-to-one   *T* is one-to-one

onto and one-to-one (II)

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation and let A be the standard matrix for T. Then:

a. T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if and only if the columns of A span $\mathbb{R}^m$;

b. T is one-to-one if and only if the columns of A are linearly independent.

onto and **one-to-one** (II)

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b. T is one-to-one if and only if the columns of A are linearly independent.

PROOF of (b.): T is one-to-one $\iff$ columns of A linearly independent.

- Let T is one-to-one $\overset{?}{\Longrightarrow}$ columns of A linearly independent.

onto and **one-to-one** (II)

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation and let A be the standard matrix for T. Then:

a. T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if and only if the columns of A span $\mathbb{R}^m$;

b. T is one-to-one if and only if the columns of A are linearly independent.

PROOF of (b.): T is one-to-one $\iff$ columns of A linearly independent.

- Let T is one-to-one $\stackrel{?}{\Longrightarrow}$ columns of A linearly independent. Proof by contradiction: if columns of A are NOT linearly independent, there would be a vector $\mathbf{x} \neq \mathbf{0}$ so that $A\mathbf{x} = \mathbf{0}$. Thus, we would have two different vectors mapped to $\mathbf{0}$:

$$A\mathbf{x} = \mathbf{0}, \quad A\mathbf{0} = \mathbf{0}, \quad \text{contradiction.}$$

Hence columns of A must be linearly independent.

PROOF of (b.): T is one-to-one $\iff$ columns of A linearly independent.

- Let columns of A linearly independent $\stackrel{?}{\Longrightarrow}$ T is one-to-one.

PROOF of (b.): T is one-to-one $\iff$ columns of A linearly independent.

- Let columns of A linearly independent $\overset{?}{\Longrightarrow}$ T is one-to-one. Proof by contradiction: if T is NOT one-to-one, there would be two vectors $\mathbf{u} \neq \mathbf{v}$ so that $A\,\mathbf{u} = A\,\mathbf{v}$. Thus, we would have a vector $\mathbf{u} - \mathbf{v} \neq \mathbf{0}$ so that

$$A\,(\mathbf{u} - \mathbf{v}) = \mathbf{0}, \qquad \text{contradiction.}$$

Hence T must be one-to-one.

§2.1 Matrix Operations

- **Notation**

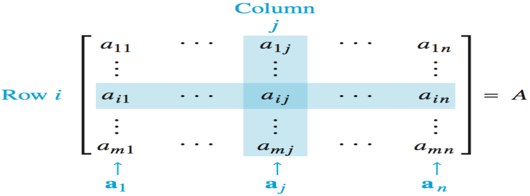
Column j

$$\text{Row } i \quad \begin{bmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix} = A$$

$$\begin{matrix} \uparrow & & \uparrow & & \uparrow \\ \mathbf{a}_1 & & \mathbf{a}_j & & \mathbf{a}_n \end{matrix}$$

§2.1 Matrix Operations

- **Notation**



- **Sum and Scalar Multiple:** Let

$$A = \begin{bmatrix} 5 & 2 & 1 \\ -7 & 1 & 2 \end{bmatrix}, \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & -4 \end{bmatrix},$$

Then $A + B = \begin{bmatrix} 6 & 4 & 4 \\ -3 & 6 & -2 \end{bmatrix}, \quad 2B = \begin{bmatrix} 2 & 4 & 6 \\ 8 & 10 & -8 \end{bmatrix},$

$$3A - 2B = \begin{bmatrix} 13 & 2 & -3 \\ -29 & -7 & 14 \end{bmatrix},$$

§2.1 Matrix Operations

- **Notation**

Column j, Row i:

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$$\uparrow \quad\quad \uparrow \quad\quad \uparrow$$
$$\mathbf{a}_1 \quad\quad \mathbf{a}_j \quad\quad \mathbf{a}_n$$

- **Sum and Scalar Multiple:**

Let A, B, and C be matrices of the same size, and let r and s be scalars.

a. $A+B=B+A$

b. $(A+B)+C=A+(B+C)$

c. $A+0=A$

d. $r(A+B)=rA+rB$

e. $(r+s)A=rA+sA$

f. $r(sA)=(rs)A$

Matrix Multiplication: Motivation

Two sets of linear equations with same coefficient matrix

$$A\mathbf{x}_1 = \mathbf{b}_1, \quad A\mathbf{x}_2 = \mathbf{b}_2, \quad \text{where}$$

$$A = \begin{bmatrix} 3 & 2 & -3 \\ 1 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix}, \quad \mathbf{b}_1 = \begin{bmatrix} 2 \\ 6 \\ 3 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}.$$

With solutions

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$$

Matrix Multiplication: Motivation

Two sets of linear equations with same coefficient matrix

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With solutions

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$$

Equations side-by-side

$$\begin{bmatrix} A\mathbf{x}_1 & A\mathbf{x}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 \end{bmatrix}, \quad (\ell)$$

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Equations side-by-side

$$\begin{bmatrix} A\mathbf{x}_1 & A\mathbf{x}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 \end{bmatrix}, \quad (\ell)$$

Define $\mathbf{X} = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 \end{bmatrix}$. Rewrite (ℓ)

$$A\mathbf{X} = \mathbf{B}, \quad \text{where} \quad A\mathbf{X} = A\begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 \end{bmatrix} \stackrel{def}{=} \begin{bmatrix} A\mathbf{x}_1 & A\mathbf{x}_2 \end{bmatrix}.$$

Matrix Multiplication (I)

In general, let $A \in \mathcal{R}^{m\times \mathbf{n}}, \qquad B \in \mathcal{R}^{\mathbf{n}\times p}$,

write B column-wise as $B = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_p \end{bmatrix}$, and define

$$A\,B = \begin{bmatrix} A\,\mathbf{b}_1 & A\,\mathbf{b}_2 & \cdots & A\,\mathbf{b}_p \end{bmatrix}.$$

Matrix Multiplication (II)

In general, let $A \in \mathcal{R}^{m\times \mathrm{n}}$, $B \in \mathcal{R}^{\mathrm{n}\times p}$, with notation

$$A = (\textbf{row} \quad i \Longrightarrow) \begin{bmatrix} * & \cdots & * & \cdots & * \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{i\mathrm{n}} \\ * & \cdots & * & \cdots & * \end{bmatrix}, \quad B = \begin{bmatrix} * & b_{1j} & * \\ \vdots & \vdots & \vdots \\ * & b_{ij} & * \\ \vdots & \vdots & \vdots \\ * & b_{\mathrm{n}j} & * \end{bmatrix}$$

where the middle column of B is marked $\overbrace{\phantom{b_{1j}}}^{\textbf{column } j}$.

$$(A\,B)_{ij} = a_{i1}\,b_{1j} + \cdots + a_{ij}\,b_{ij} + \cdots + a_{i\mathrm{n}}\,b_{\mathrm{n}j}.$$

Matrix Multiplication (II)

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$$(A\,B)_{ij} = a_{i1}\,b_{1j} + \cdots + a_{ij}\,b_{ij} + \cdots + a_{i\mathrm{n}}\,b_{\mathrm{n}j}.$$

Example:

$$\rightarrow \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{bmatrix} = \begin{bmatrix} \square & \square & 2(6)+3(3) \\ \square & \square & \square \end{bmatrix} = \begin{bmatrix} \square & \square & 21 \\ \square & \square & \square \end{bmatrix}$$

Matrix Multiplication (II)

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Example:

$$\begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{bmatrix} = \begin{bmatrix} \square & \square & 21 \\ \square & 1(3)+-5(-2) & \square \end{bmatrix} = \begin{bmatrix} \square & \square & 21 \\ \square & 13 & \square \end{bmatrix}$$

Matrix Multiplication (III)

Theorem: Let $A \in \mathcal{R}^{m \times \mathrm{n}}, \quad B \in \mathcal{R}^{\mathrm{n} \times t}, \quad B \in \mathcal{R}^{t \times p},$

Then $\quad A\,(B\,C) = (A\,B)\;C.$

PROOF: Write C column-wise $\mathbf{C} = \left[\begin{array}{cccc} \mathbf{c}_1 & \mathbf{c}_2 & \cdots & \mathbf{c}_p \end{array}\right]$. Then

$$
\begin{aligned}
A\,(B\,C) &= A\left(B\left[\begin{array}{cccc} \mathbf{c}_1 & \mathbf{c}_2 & \cdots & \mathbf{c}_p \end{array}\right]\right) \\
&= A\left(\left[\begin{array}{cccc} B\,\mathbf{c}_1 & B\,\mathbf{c}_2 & \cdots & B\,\mathbf{c}_p \end{array}\right]\right) \\
&= \left[\begin{array}{cccc} A\,(B\,\mathbf{c}_1) & A\,(B\,\mathbf{c}_2) & \cdots & A\,(B\,\mathbf{c}_p) \end{array}\right]
\end{aligned}
$$

Matrix Multiplication (III)

$$\textbf{Theorem: } \text{Let } A \in \mathcal{R}^{m\times \mathsf{n}}, \quad B \in \mathcal{R}^{\mathsf{n}\times t}, \quad B \in \mathcal{R}^{t\times p},$$

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PROOF: Write C column-wise $\mathbf{C} = \begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \cdots & \mathbf{c}_p \end{bmatrix}$. Then

$$\begin{aligned}
A\,(B\,C) &= A\left(B\begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \cdots & \mathbf{c}_p \end{bmatrix}\right) \\
&= A\left(\begin{bmatrix} B\,\mathbf{c}_1 & B\,\mathbf{c}_2 & \cdots & B\,\mathbf{c}_p \end{bmatrix}\right) \\
&= \begin{bmatrix} A\,(B\,\mathbf{c}_1) & A\,(B\,\mathbf{c}_2) & \cdots & A\,(B\,\mathbf{c}_p) \end{bmatrix} \\
&\underline{\underline{\text{Book Def}}} \begin{bmatrix} (A\,B)\,\mathbf{c}_1 & (A\,B)\,\mathbf{c}_2 & \cdots & (A\,B)\,\mathbf{c}_p \end{bmatrix}
\end{aligned}$$

Matrix Multiplication (III)

Theorem: Let $A \in \mathcal{R}^{m\times \mathrm{n}}, \quad B \in \mathcal{R}^{\mathrm{n}\times t}, \quad B \in \mathcal{R}^{t\times p},$

$$\text{Then} \quad A\,(B\,C) = (A\,B)\;C.$$

PROOF: Write C column-wise $\mathbf{C} = \left[\begin{array}{cccc} \mathbf{c}_1 & \mathbf{c}_2 & \cdots & \mathbf{c}_p \end{array}\right]$. Then

$$\begin{array}{rcl}
A\,(B\,C) & = & A\left(B\left[\begin{array}{cccc} \mathbf{c}_1 & \mathbf{c}_2 & \cdots & \mathbf{c}_p \end{array}\right]\right) \\
& = & A\left(\left[\begin{array}{cccc} B\,\mathbf{c}_1 & B\,\mathbf{c}_2 & \cdots & B\,\mathbf{c}_p \end{array}\right]\right) \\
& = & \left[\begin{array}{cccc} A\,(B\,\mathbf{c}_1) & A\,(B\,\mathbf{c}_2) & \cdots & A\,(B\,\mathbf{c}_p) \end{array}\right] \\
& \underline{\underline{\text{Book Def}}} & \left[\begin{array}{cccc} (A\,B)\;\mathbf{c}_1 & (A\,B)\;\mathbf{c}_2 & \cdots & (A\,B)\;\mathbf{c}_p \end{array}\right] \\
& = & (A\,B)\left[\begin{array}{cccc} \mathbf{c}_1 & \mathbf{c}_2 & \cdots & \mathbf{c}_p \end{array}\right] \\
& = & (A\,B)\;C.
\end{array}$$

WARNING: $AB \neq BA$ in general

$$\text{Let } A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix}.$$

$$\text{Then } AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad BA = \begin{bmatrix} 6 & 12 \\ -3 & -6 \end{bmatrix}.$$

Matrix Transpose

Transpose of matrix A is denoted A^T, and formed by setting each column in A^T from corresponding row in A.

$$\text{Let } A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}.$$

$$\text{Then } A^T = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}, \quad B^T = \begin{bmatrix} 2 & -1 \\ 2 & -1 \\ 1 & 0 \end{bmatrix}.$$

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$$\text{Then } A^T = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}, \quad B^T = \begin{bmatrix} 2 & -1 \\ 2 & -1 \\ 1 & 0 \end{bmatrix}.$$

Theorem: $\left(A^T\right)^T = A, \qquad (A\,B)^T = B^T\,A^T.$