Self Introduction

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- **Office Hours:** MTuWF 4:15-5:30PM
- **Class Website:** math.berkeley.edu/∼mgu/MA54Spring2019
- Lecture notes available on bCourses

Text Book

- UC Berkeley Edition,
  Linear Algebra and Applications, required.
- Closely follow Math Dept outlines for Math 54.
- Students responsible for material left out in lectures.

Math 54

Math 54 - Linear Algebra & Differential Equations -- [4 units]

Course Format: Three hours of lecture and three hours of discussion per week.

Prerequisites: 1A-1B, 10A-10B or equivalent.

Description: Basic linear algebra; matrix arithmetic and determinants. Vector spaces; inner product spaces. Eigenvalues and eigenvectors; linear transformations, symmetric matrices. Linear ordinary differential equations (ODE); systems of linear ODE. Fourier series. (F,SP)

Textbook: Lay-Lay-McDonald, **Linear Algebra and its Applications** (5th ed) and Nagle-Saff-Snider, **Fundamentals of Differential Equations and Boundary Value Problems** (6th ed). A specially priced UC Berkeley paperback edition of both books is available.

Part One: (Lay et al)

Chapter 1: Linear Equations in linear algebra	4 hours
Sections 1.1-1.5, 1.7-1.9	
Chapter 2: Matrix Algebra	2 hours
Sections 2.1-2.3	
Chapter 3: Determinants	3 hours
Sections 3.1-3.3	
Chapter 4: Vector Spaces	5 hours
Sections 4.1-4.7	
Chapter 5: Eigenvalues and eigenvectors	4 hours
Sections 5.1-5.4	
Chapter 6: Orthogonality, least Squares	5 hours
Sections 6.1-6.5, 6.7	
Chapter 7: Symmetric matrices, applications	3 hours
Sections 7.1, 7.3-7.5	

Class Work

- Weekly home work sets;
  Count best 10, total 15 points.
- Weekly Quizzes (except on midterm weeks);
  Count best 10, total 15 points.
- Two midterm exams:
  the worse is 15 points, the better 25 points.
- 1 final exam, 30 points.

Class Work

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  Count best 10, total 15 points.
- Weekly Quizzes (except on midterm weeks);
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- Two midterm exams:
  the worse is 15 points, the better 25 points.
- 1 final exam, 30 points.
- **If you miss one midterm, your other midterm and the final will be worth 30 and 40 points, respectively.**

Exam Schedule

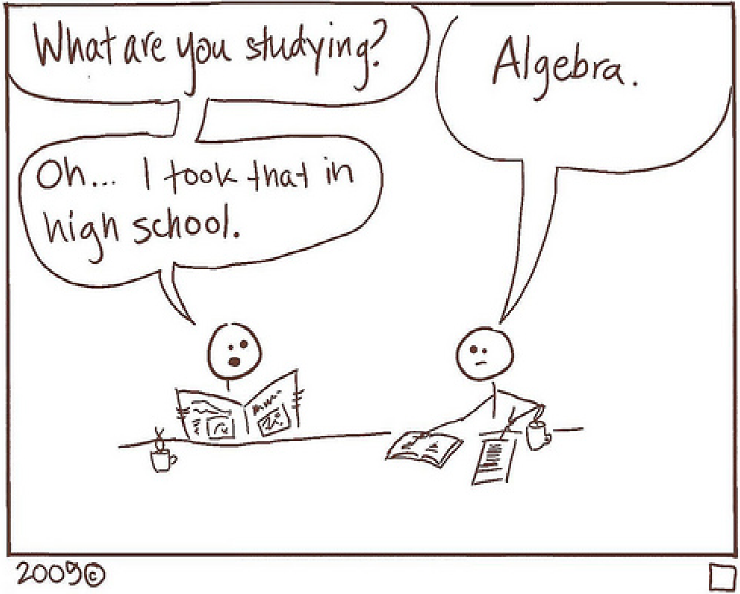
- **Midterm I**: Feb. 20 (Wed.) in class
- **Midterm II**: Mar. 20 (Wed.) in class
- **Final Exam**: Tue., May 14, 7:00–10:00pm

Grade Scale

- **A**- to **A+**: at least 85 points;
- **B**- to **B+**: between 70 and 85 points;
- **C**- to **C+**: between 60 and 70 points;
- **D**: between 55 and 60 points;
- **F**: less than 55 points.

No grade curve; most people get *A* level or *B* level grades.

I took Algebra in High School



§1.1 Systems of Linear Equations

linear equation is of form

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = b, \quad \text{where}$$

- $a_1, a_2, \cdots, a_n$ are COEFFICIENTS;
- $x_1, x_2, \cdots, x_n$ are VARIABLES;
- b is RIGHT HAND SIDE.

Linear equation example:

$$4\,x_1 - 5\,x_2 + 3\,x_3 = 2.$$

System of linear equations example:

$$\begin{aligned} 4\,x_1 - 5\,x_2 + 3\,x_3 &= 2, \\ 2\,x_2 - x_3 &= 1, \\ -3\,x_1 + 8\,x_3 &= 5. \end{aligned}$$

Linear Algebra is 2000 years old (I)

250 AD, *9 Chapters of the Mathematical Art*

Mathematician using counting rods on a counting table

Liu Hui said *Nine Chapters* were already old (≈ 100 BC ?) when he wrote about them in 250 AD.

The *only* treatment of linear eqns. and Gaussian elimination in antiquity.

Linear Algebra is 2000 years old (II)

Chapter 8 of the *Nine Chapters*

Problem 1 (of 18 similar problems)

For 3 sheaves of rice from top-grade rice paddies,
and 2 sheaves from medium-grade paddies,
and 1 sheaf from low-grade paddies,
the combined yield is 39 dou of grain.

And so on for two more collections of paddies.

For each grade, 1 sheaf yields how much rice?

Counting Table Setup

1	2	3	top-grade sheaves
2	3	2	mid-grade
3	1	1	low-grade
26	34	39	grain

Linear Algebra is 2000 years old (III): One dou of grain

Linear Algebra is 2000 years old (IV)

Chapter 8 of the *Nine Chapters*

<u>Solution</u>

$\times 3 \quad - \times 1$

$\times 3 - \times 2$

1	2	3
2	3	2
3	1	1
26	34	39

$\Rightarrow$ $\times 5 - \times 4$

		3
4	5	2
8	1	1
39	24	39

$\Rightarrow$ $\div 9$

		3
	5	2
36	1	1
99	24	39

$\Rightarrow$

		3
	5	2
4	1	1
11	24	39

$\cdots \Rightarrow$

		4
	4	
4		
11	17	37

$\Rightarrow$ low $= \frac{11}{4}$, mid $= \frac{17}{4}$, top $= \frac{37}{4}$

R. Hart, *The Chinese Roots of Linear Algebra*, Johns-Hopkins University Press, 2011.

Reviewed by J. F. Grcar, *Bull. Amer. Math. Soc.* 49 (2012), 585-590.

Linear Algebra is 2000 years old, but named after Gauss

Successful Calculations Have 4 Ingredients

C. F. Gauss
1777–1855

1. Impact for society
2. Mathematical formulation
3. Technology for computing
4. Algorithm appropriate to # 3

1	impact	astronomy and cartography
2	math	Gauss's adjustment of observations
3	technology	hand computing (probably logarithms)
4	algorithm	*Gauss's brackets*

Linear equations, exactly one solution

System of two linear equations:

$$\begin{aligned} x_1 - 2\,x_2 &= -1, \quad (\ell_1) \\ -x_1 + 3\,x_2 &= 3. \quad (\ell_2) \end{aligned}$$

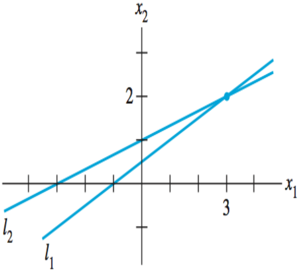


FIGURE 1 Exactly one solution.

Linear equations, no solution or too many solutions

(a) $$\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 2x_2 &= 3 \end{aligned}$$

(b) $$\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 2x_2 &= 1 \end{aligned}$$

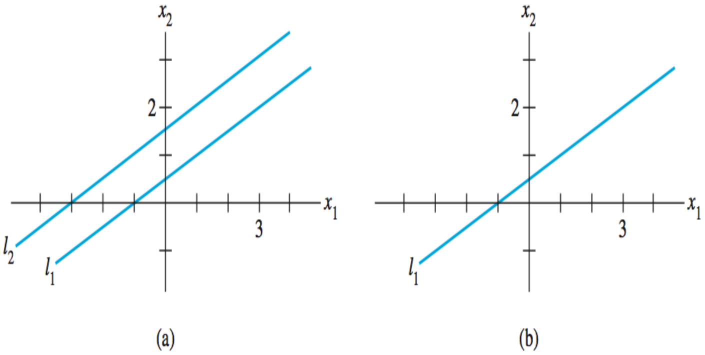


FIGURE 2 (a) No solution. (b) Infinitely many solutions.

Linear equations, solution in 3D

System of linear equations:
$$\begin{aligned} x_1 - 2\,x_2 + x_3 &= 0, \\ x_2 - 8\,x_3 &= 8, \\ 5\,x_1 - 5\,x_3 &= 10. \end{aligned}$$

Solution $x_1 = 1$, $x_2 = 0$, $x_3 = -1$.

Linear equations, solution in 3D

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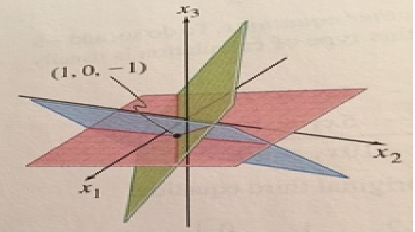

Each of the original equations determines a plane in three-dimensional space. The point (1, 0, −1) lies in all three planes.

Matrix Notation

System of linear equations:

$$\begin{aligned} x_1 - 2\,x_2 + x_3 &= 0, \\ x_2 - 8\,x_3 &= 8, \\ 5\,x_1 - 5\,x_3 &= 10. \end{aligned}$$

Matrix Notation

System of linear equations:

$$\begin{aligned} x_1 - 2\,x_2 + x_3 &= 0, \\ x_2 - 8\,x_3 &= 8, \\ 5\,x_1 - 5\,x_3 &= 10. \end{aligned}$$

- **coefficient matrix**

$$A = \begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & -8 \\ 5 & 0 & -5 \end{pmatrix}$$

- **right hand side** (RHS)

$$b = \begin{pmatrix} 0 \\ 8 \\ 10 \end{pmatrix}$$

- **augmented matrix**

$$\mathcal{A} = \left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{array}\right) = \left(\begin{array}{c|c} A & b \end{array}\right)$$

Nine Chapters Problem

System of linear equations:

$$\begin{aligned} x_1 + 2\,x_2 + 3\,x_3 &= 26, & (\ell_1)\\ 2\,x_1 + 3\,x_2 + x_3 &= 34, & (\ell_2)\\ 3\,x_1 + 2\,x_2 + x_3 &= 39. & (\ell_3) \end{aligned}$$

$x_1 = \textbf{top}-\textbf{grade}$, $x_2 = \textbf{mid}-\textbf{grade}$, $x_3 = \textbf{low}-\textbf{grade}$.

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$x_1 =$ **top−grade**, $x_2 =$ **mid−grade**, $x_3 =$ **low−grade**.

- **coefficient matrix**
$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 2 & 1 \end{pmatrix}$$
- **right hand side** (RHS)
$$b = \begin{pmatrix} 26 \\ 34 \\ 39 \end{pmatrix}$$
- **augmented matrix**
$$\mathcal{A} = \left(\begin{array}{ccc|c} 1 & 2 & 3 & 26 \\ 2 & 3 & 1 & 34 \\ 3 & 2 & 1 & 39 \end{array}\right) = \left(\begin{array}{c|c} A & b \end{array} \right)$$

Gaussian Elimination (I)

- $3 \times (\ell_1) - 1 \times (\ell_3) \to$ **new** (ℓ_1), $3 \times (\ell_2) - 2 \times (\ell_3) \to$ **new** (ℓ_2)

System of linear equations:

$$\begin{aligned} 4\,x_2 + 8\,x_3 &= 39, \quad (\ell_1) \\ 5\,x_2 + x_3 &= 24, \quad (\ell_2) \\ 3\,x_1 + 2\,x_2 + x_3 &= 39. \quad (\ell_3) \end{aligned}$$

- **augmented matrix**

$$\mathcal{A} = \left(\begin{array}{ccc|c} 0 & 4 & 8 & 39 \\ 0 & 5 & 1 & 24 \\ 3 & 2 & 1 & 39 \end{array} \right)$$

Gaussian Elimination (II)

- $5 \times (\ell_1) - 4 \times (\ell_2) \to$ **new** (ℓ_1)

$$\text{System of linear equations:} \quad \begin{aligned} 36\,x_3 &= 99, & (\ell_1) \\ 5\,x_2 + x_3 &= 24, & (\ell_2) \\ 3\,x_1 + 2\,x_2 + x_3 &= 39. & (\ell_3) \end{aligned}$$

- **augmented matrix**

$$\mathcal{A} = \left(\begin{array}{ccc|c} 0 & 0 & 36 & 99 \\ 0 & 5 & 1 & 24 \\ 3 & 2 & 1 & 39 \end{array} \right)$$

Solution

$$x_3 = \frac{11}{4}, \quad x_2 = \frac{17}{4}, \quad x_1 = \frac{37}{4}.$$

ELEMENTARY ROW OPERATIONS

1. (Replacement) Replace one row by the sum of itself and a multiple of another row.[2]
2. (Interchange) Interchange two rows.
3. (Scaling) Multiply all entries in a row by a nonzero constant.

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- Example: **Row interchange**

System of linear equations:

$$
\begin{aligned}
36\,x_3 &= 99, \quad (\ell_1)\\
5\,x_2 + x_3 &= 24, \quad (\ell_2)\\
3\,x_1 + 2\,x_2 + x_3 &= 39. \quad (\ell_3)
\end{aligned}
$$

$$\Downarrow$$

New System:

$$
\begin{aligned}
3\,x_1 + 2\,x_2 + x_3 &= 39, \quad (\ell_1)\\
5\,x_2 + x_3 &= 24, \quad (\ell_2)\\
36\,x_3 &= 99. \quad (\ell_3)
\end{aligned}
$$

TWO FUNDAMENTAL QUESTIONS ABOUT A LINEAR SYSTEM

1. Is the system consistent; that is, does at least one solution *exist*?
2. If a solution exists, is it the *only* one; that is, is the solution *unique*?

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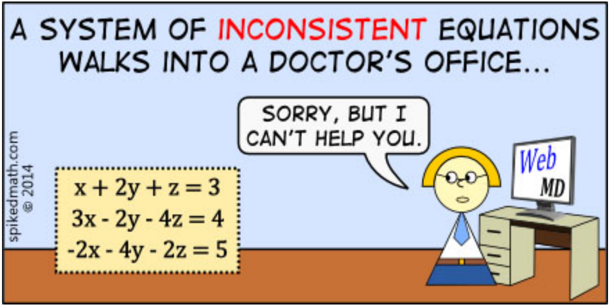

Inconsistent Equations

System of linear equations:

$$\begin{aligned} x + 2\,y + z &= 3, \quad (\ell_1) \\ 3\,x - 2\,y - 4\,z &= 4, \quad (\ell_2) \\ -2\,x - 4\,y - 2\,z &= 5. \quad (\ell_3) \end{aligned}$$

Inconsistent Equations

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- **augmented matrix**

$$\mathcal{A} = \left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 3 & -2 & -4 & 4 \\ -2 & -4 & -2 & 5 \end{array} \right) = \left(\begin{array}{c|c} A & b \end{array} \right)$$

- $1 \times (\ell_2) - 3 \times (\ell_1) \rightarrow$ **new** (ℓ_2),
  $1 \times (\ell_3) - (-2) \times (\ell_3) \rightarrow$ **new** (ℓ_3).
- new **augmented matrix**

$$\mathcal{A} = \left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -8 & -7 & -5 \\ 0 & 0 & 0 & 11 \end{array} \right)$$

§1.2 Row Reduction and Echelon Form (I)

In Augmented matrix, the **row echelon form** (REF)

$$\mathcal{A} = \left(\begin{array}{c|c} A & b \end{array}\right)$$

$$\begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * \end{bmatrix}$$

where $\blacksquare \neq 0, * =$ any entry.
REF allows $\blacksquare$ variables to be easily solved.

§1.2 Row Reduction and Echelon Form (II)

In rectangular matrix, a **leading entry** of a row is the leftmost non-zero.
For augmented matrix in row echelon form

$$\begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * \end{bmatrix}$$

variables with leading entry $\blacksquare$ can be solved in terms of the other variables.

§1.2 Row Reduction and Echelon Form (III)

A rectangular matrix is in **echelon form** (or **row echelon form**) if it has the following three properties:

1. All nonzero rows are above any rows of all zeros.

2. Each leading entry of a row is in a column to the right of the leading entry of the row above it.

3. All entries in a column below a leading entry are zeros.

$$\begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * \end{bmatrix}$$

§1.2 Row Reduction and Echelon Form (IV)

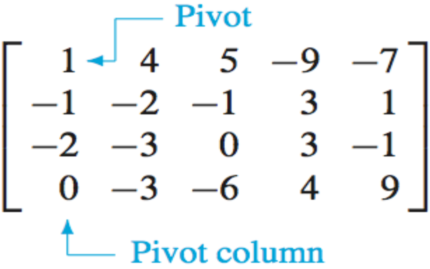
A rectangular matrix is in **echelon form** (or **row echelon form**) if it has the following three properties:

1. All nonzero rows are above any rows of all zeros.
2. Each leading entry of a row is in a column to the right of the leading entry of the row above it.
3. All entries in a column below a leading entry are zeros.

$$\begin{bmatrix} 1 & 0 & -5 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Row Reduction Algorithm (I)

In rectangular matrix, a **pivot position** is a position that corresponds to a leading entry; and **pivoting column** is a column that contains a pivot position.



$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

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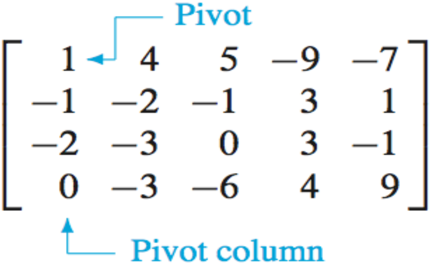

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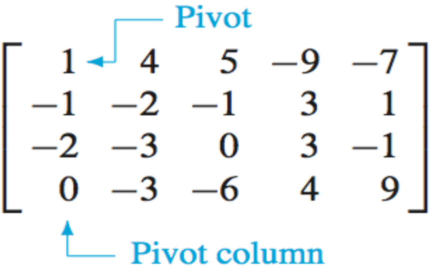


$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

Row Reduction Algorithm (II)

$$A = \begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$$

Row Reduction Algorithm (III)

Interchange rows 1 and 3 to reach pivot position.

Pivot

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

Pivot column

Row Reduction Algorithm (IV)

$$
\begin{aligned}
\textbf{row}_2 - (-1) \times \textbf{row}_1 &\rightarrow \textbf{row}_2, \\
\textbf{row}_3 - (-2) \times \textbf{row}_1 &\rightarrow \textbf{row}_3
\end{aligned}
$$

Pivot

$$
\begin{bmatrix}
1 & 4 & 5 & -9 & -7 \\
0 & 2 & 4 & -6 & -6 \\
0 & 5 & 10 & -15 & -15 \\
0 & -3 & -6 & 4 & 9
\end{bmatrix}
$$

Next pivot column

no row interchange needed

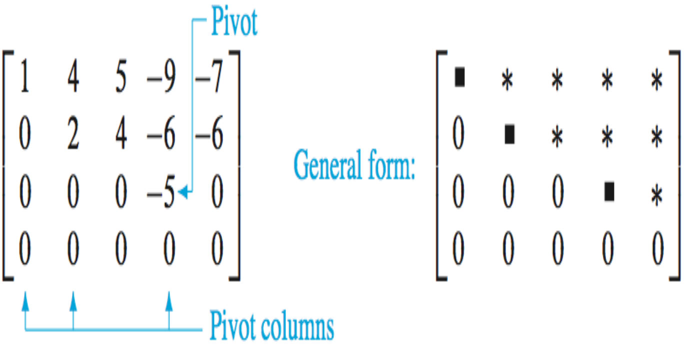
Row Reduction Algorithm (V)

$$\begin{aligned} \mathbf{row}_3 - \left(\frac{5}{2}\right) \times \mathbf{row}_2 \quad &\rightarrow \quad \mathbf{row}_3, \\ \mathbf{row}_4 - \left(\frac{-3}{2}\right) \times \mathbf{row}_2 \quad &\rightarrow \quad \mathbf{row}_4 \end{aligned}$$

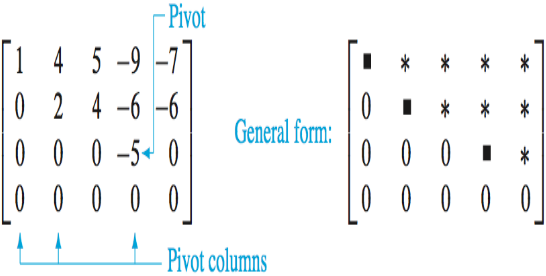
$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix}$$

interchange $\mathbf{row}_3$ and $\mathbf{row}_4$

Row Reduction Algorithm (VI)



Solution of Linear Equations



$$
\begin{aligned}
x_1 + 4\,x_2 + 5\,x_3 - 9x_4 &= -7, \\
2\,x_2 + 4\,x_3 - 6x_4 &= -6, \\
-5x_4 &= 0.
\end{aligned}
$$

Solution: with x_3 free,

$$x_4 = 0,\;\; x_2 = -3 - 2\,x_3,\;\; x_1 = 5 + 3\,x_3.$$

x_1, x_2, x_4: **basic variables**; x_3: **free variable.**

- **Existence**: A linear system is consistent $\iff$ rightmost column in $(A \mid b)$ is *not* pivot column.
- **Uniqueness**: A consistent linear system contains
  - either: *unique* solution but no free variable,
  - or: *infinitely* many solutions with at least one free variable.

§1.3 Vector Equations

Vectors in $\mathcal{R}^2$

$$\mathbf{u}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \ \mathbf{u}_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \ \mathbf{u}_3 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \ \mathbf{u}_4 = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}.$$

Then

$$\mathbf{u}_1 + \mathbf{u}_2 = \begin{pmatrix} 5 \\ 2 \end{pmatrix}, \ \mathbf{u}_3 = \mathbf{u}_1, \ 2\,\mathbf{u}_4 = \begin{pmatrix} 2\,\mu_1 \\ 2\,\mu_2 \end{pmatrix}.$$

§1.3 Vector Equations

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Then

$$\mathbf{u}_1 + \mathbf{u}_2 = \begin{pmatrix} 5 \\ 2 \end{pmatrix}, \ \mathbf{u}_3 = \mathbf{u}_1, \ 2\,\mathbf{u}_4 = \begin{pmatrix} 2\,\mu_1 \\ 2\,\mu_2 \end{pmatrix}.$$

Vectors in $\mathcal{R}^3$: $\quad \mathbf{a} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix},$

Vectors in $\mathcal{R}^n$: $\quad \mathbf{w} = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_n \end{pmatrix}.$

Geometric Description in $\mathcal{R}^2$

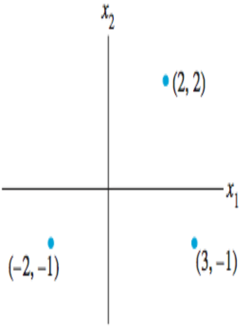


FIGURE 1 Vectors as points.

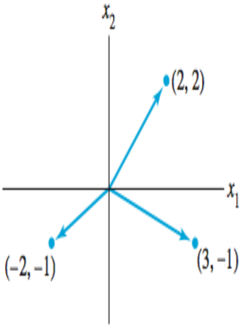
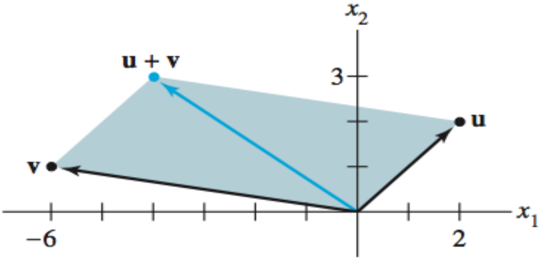


FIGURE 2 Vectors with arrows.

Parallelogram Rule

Vectors in $\mathcal{R}^2$

$$\mathbf{u} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \ \mathbf{v} = \begin{pmatrix} -6 \\ 1 \end{pmatrix}, \ \mathbf{u} + \mathbf{v} = \begin{pmatrix} -4 \\ 3 \end{pmatrix}.$$



Linear Combinations

Given vectors $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n \in \mathcal{R}^n$, and scalars $c_1, c_2, \cdots, c_n \in \mathcal{R}$, the vector

$$\mathbf{y} = c_1\,\mathbf{u}_1 + c_2\,\mathbf{u}_2 + \cdots + c_n\,\mathbf{u}_n$$

is a **Linear Combination** of vectors $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ with **weights** $c_1, c_2, \cdots, c_n$.

Example: With $c_1 = -2, c_2 = 3$,

$$\mathbf{u}_1 = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}, \; \mathbf{u}_2 = \begin{pmatrix} 5 \\ 2 \\ 1 \end{pmatrix}.$$

$$\mathbf{y} = c_1\,\mathbf{u}_1 + c_2\,\mathbf{u}_2 = \begin{pmatrix} 11 \\ 8 \\ 3 \end{pmatrix}.$$

Span (I)

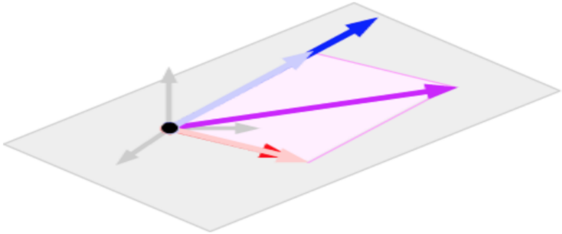
If $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are in $\mathbb{R}^n$, then the set of all linear combinations of $\mathbf{v}_1, \ldots, \mathbf{v}_p$ is denoted by Span $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ and is called the **subset of $\mathbb{R}^n$ spanned** (or **generated**) **by** $\mathbf{v}_1, \ldots, \mathbf{v}_p$. That is, Span $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is the collection of all vectors that can be written in the form

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p$$

with $c_1, \ldots, c_p$ scalars.

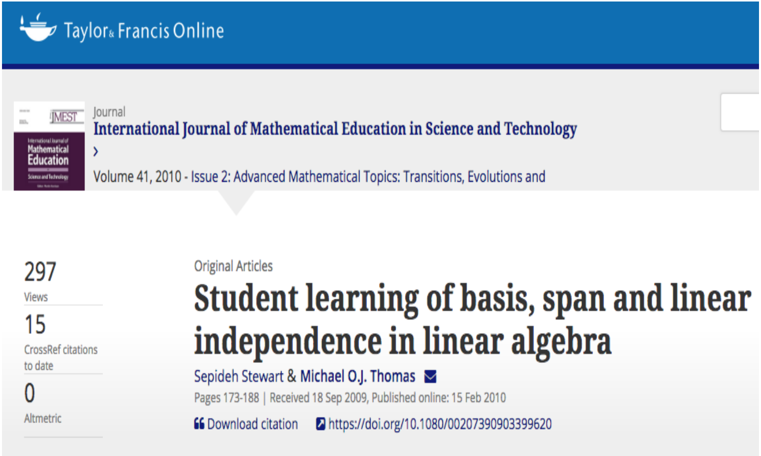
Span (II)

Example span in $\mathcal{R}^3$



To **Span** or not to **Span?**

Linear Algebra is both interesting and challenging



Taylor & Francis Online

Journal

International Journal of Mathematical Education in Science and Technology

Volume 41, 2010 - Issue 2: Advanced Mathematical Topics: Transitions, Evolutions and

297 Views

15 CrossRef citations to date

0 Altmetric

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Student learning of basis, span and linear independence in linear algebra

Sepideh Stewart & Michael O.J. Thomas

Pages 173-188 | Received 18 Sep 2009, Published online: 15 Feb 2010

Download citation https://doi.org/10.1080/00207390903399620

§1.4 Matrix Equation $A\mathbf{x} = \mathbf{b}$

Column vector representation of matrix $A \in \mathcal{R}^{m\times n}$:

$$A = \left(\begin{array}{r|r|r|l} 1 & 2 & 1 & 3 \\ 3 & -2 & -4 & 4 \\ -2 & -4 & -2 & 5 \end{array}\right) \in \mathcal{R}^{3\times 4}$$

$$\Uparrow \quad \Uparrow \quad \Uparrow \quad \Uparrow$$

$$= \left(\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3 \quad \mathbf{a}_4 \right), \quad \text{where} \quad \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4 \in \mathcal{R}^3$$

Matrix-vector Product $\Longleftrightarrow$ Linear Combination (I)

If A is an $m \times n$ matrix, with columns $\mathbf{a}_1, \ldots, \mathbf{a}_n$, and if $\mathbf{x}$ is in $\mathbb{R}^n$, then the **product of A and $\mathbf{x}$**, denoted by $A\mathbf{x}$, is **the linear combination of the columns of A using the corresponding entries in $\mathbf{x}$ as weights**; that is,

$$A\mathbf{x} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n$$

Matrix-vector Product $\Longleftrightarrow$ Linear Combination (II)

Example:

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 2 \\ -5 \end{bmatrix} + 7 \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ 0 \end{bmatrix} + \begin{bmatrix} 6 \\ -15 \end{bmatrix} + \begin{bmatrix} -7 \\ 21 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

Linear Equations in terms of Matrix-vector Product

Example

$$x_1 + 2x_2 - x_3 = 4$$

$$-5x_2 + 3x_3 = 1$$

equivalent to

$$x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ -5 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

Linear Equations in terms of Matrix-vector Product

Example

$$\begin{aligned} x_1 + 2x_2 - \ x_3 &= 4 \\ -5x_2 + 3x_3 &= 1 \end{aligned}$$

equivalent to

$$x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ -5 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

Linear Equations in terms of Linear Combinations

If A is an $m \times n$ matrix, with columns $\mathbf{a}_1, \ldots, \mathbf{a}_n$, and if $\mathbf{b}$ is in $\mathbb{R}^m$, the matrix equation

$$A\mathbf{x} = \mathbf{b} \tag{4}$$

has the same solution set as the vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b} \tag{5}$$

Linear Equations in terms of Linear Combinations

If A is an $m \times n$ matrix, with columns $\mathbf{a}_1, \ldots, \mathbf{a}_n$, and if $\mathbf{b}$ is in $\mathbb{R}^m$, the matrix equation

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has the same solution set as the vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b} \tag{5}$$

The equation $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $\mathbf{b}$ is a linear combination of the columns of A.