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Math54 Sample Midterm II, Spring 2019

This is a closed everything exam, except a standard one-page cheat sheet (on one-side only). You need to justify every one of your answers. Completely correct answers given without justification will receive little credit. Problems are not necessarily ordered according to difficulties. You need not simplify your answers unless you are specifically asked to do so.

Problem	Maximum Score	Your Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total	100	

Write your personal information below.

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Your GSI: DongGyu Lim

Your SID: N/A

1. Consider the following map from $\mathbf{R}^2$ to $\mathbf{R}^2$:

$$\mathbf{T}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y-1 \\ x-y+1 \end{pmatrix}.$$

Is $\mathbf{T}$ a linear transform? Explain.

One property of a linear transformation is that it sends the zero vector to the zero vector. However, $T\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0+0-1 \\ 0-0+1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \neq$ the zero vector.

So, T is not a linear transformation.

Explanation why a linear transformation sends the zero vector to the zero vector

: T should satisfy $T(c \cdot v) = c \cdot T(v)$ for any $c \in \mathbb{R}$, $v \in \mathbb{R}^2$

Plug in $c=0$, then $T(0 \cdot v) = 0\, T(v)$

, that is, $T(\vec{0}) = \vec{0}$.

2. Let

$$\mathbf{u} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \quad A = \mathbf{u}\,\mathbf{v}^T,$$

where $\alpha, \beta \in \mathcal{R}$.

- What are the dimensions of matrix A?
- Explain that $\mathbf{rank}\,(A) \leq 1$ for any $\alpha, \beta \in \mathcal{R}$.
- Explain conditions under which $\mathbf{rank}\,(A) = 0$.

- u is 3×1. v is 2×1. So, v^T is 1×2. Now, $u \cdot v^T$ is 3×2.
- First of all. $A = \begin{bmatrix} \alpha & \beta \\ 2\alpha & 2\beta \\ 3\alpha & 3\beta \end{bmatrix}$. So, $\text{Col } A = \text{Span}\left\{\begin{bmatrix} \alpha \\ 2\alpha \\ 3\alpha \end{bmatrix}, \begin{bmatrix} \beta \\ 2\beta \\ 3\beta \end{bmatrix}\right\}$.

$$= \text{Span}\left\{\alpha \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \beta \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right\} \subseteq \text{Span}\left\{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right\}$$

(this is because $\alpha \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \beta \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \in \text{Span}\left\{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right\}$)

So, $\text{rank}(A) = \dim \text{Col } A \leq \dim \text{Span}\left\{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right\} = 1$.

- $\text{rank}(A) = 0$ means $\dim \text{Col } A = 0$. The only zero-dimensional space is the zero space (that is, there is only one vector (the zero vector)). Therefore, $\alpha \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\beta \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ all need to be the zero vector. $\therefore\ \alpha, \beta = 0$.

3. Let $A = \begin{pmatrix} 3 & 4 \\ \alpha & 3 \end{pmatrix}$.

- Explain why the matrix A is diagonalizable for $\alpha > 0$.
- is the matrix A diagonalizable for $\alpha = 0$?

- The characteristic polynomial $\chi_A(\lambda)$ of A is

$$\det \begin{bmatrix} 3-\lambda & 4 \\ \alpha & 3-\lambda \end{bmatrix} = (\lambda-3)^2 - 4\alpha.$$

So, the characteristic equation is $(\lambda-3)^2 = 4\alpha$.

Suppose $\alpha > 0$, then $(\lambda-3)^2 = 4\alpha$ has two distinct roots.

: $3+2\sqrt{\alpha}$, $3-2\sqrt{\alpha}$.

For ~~each~~ distinct eigenvalues, we have linearly independent eigenvectors.

$\mathbb{R}^2$ is of dimension 2, so we will have a basis consisting of eigenvectors. So, A is diagonalizable

- For $\alpha = 0$, $\chi_A(\lambda) = (\lambda-3)^2$. So, $\lambda = 3$ is the only eigenvalue.

Now, we need to check if the eigenspace corresponding to 3 has dimension 2 or less.

$$\mathrm{Nul}(A-3I) = \mathrm{Nul}\begin{bmatrix} 0 & 4 \\ 0 & 0 \end{bmatrix}$$

Using the rank theorem, $\dim \mathrm{Nul}\begin{bmatrix} 0 & 4 \\ 0 & 0 \end{bmatrix} = 2 - \dim \mathrm{Col}\begin{bmatrix} 0 & 4 \\ 0 & 0 \end{bmatrix}$

$= 2 - 1 = 1.$

So, the dimension is less than 2. Therefore, A is not diagonalizable.

4. Let

$$A = \begin{pmatrix} 1 & 1 \\ 2 & -2 \\ 3 & 1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

Find the least squares solution for the problem

$$\min_{\mathbf{x}} \|A\,\mathbf{x} - \mathbf{b}\|.$$

The least squares solution is the same as the solution of the normal equation

$$A^T A \mathbf{x} = A^T b.$$

$$A^T A = \begin{bmatrix} 14 & 0 \\ 0 & 6 \end{bmatrix} \text{ and } A^T b = \begin{bmatrix} 4 \\ 2 \end{bmatrix}.$$

Put $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. Then, it gives $14x_1 = 4$, $6x_2 = 2$. So, $x_1 = 2/7$, $x_2 = 1/3$.

The least squares solution is $\begin{bmatrix} 2/7 \\ 1/3 \end{bmatrix}$.

5. Let

$$V = \text{span}\left(\begin{pmatrix}1\\1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\-1\\-1\\1\end{pmatrix}\right).$$

Find an orthonormal basis for the orthogonal complement of V.

The orthogonal complement of V can be computed as the null space of $\begin{bmatrix}1&1&1&1\\1&-1&-1&1\end{bmatrix}$. By row reducing, one get $\begin{bmatrix}1&0&0&1\\0&1&1&0\end{bmatrix}$.

Hence, we can let x_3 and x_4 be free and $X_1+X_4=0$, $X_2+X_3=0$

So, $\text{Nul}\begin{bmatrix}1&1&1&1\\1&-1&-1&1\end{bmatrix} = \left\{\begin{bmatrix}X_1\\X_2\\X_3\\X_4\end{bmatrix} : \begin{matrix}X_1=-X_4\\X_2=-X_3\end{matrix} \text{ and } X_3, X_4 : \text{free}\right\}$

$$V^\perp = \left\{\begin{bmatrix}-X_4\\-X_3\\X_3\\X_4\end{bmatrix} : X_3, X_4 \in \mathbb{R}\right\}$$

$$= \left\{X_3\begin{bmatrix}0\\-1\\1\\0\end{bmatrix} + X_4\begin{bmatrix}-1\\0\\0\\1\end{bmatrix} : X_3, X_4 \in \mathbb{R}\right\} = \text{Span}\left\{\begin{bmatrix}0\\-1\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\0\\1\end{bmatrix}\right\}$$

Fortunately, $\begin{bmatrix}0\\-1\\1\\0\end{bmatrix}$ and $\begin{bmatrix}-1\\0\\0\\1\end{bmatrix}$ are orthogonal. We only need to normalize the length to find an orthonormal basis.

Answer: $\left\{\begin{bmatrix}0\\-1/\sqrt{2}\\1/\sqrt{2}\\0\end{bmatrix}, \begin{bmatrix}-1/\sqrt{2}\\0\\0\\1/\sqrt{2}\end{bmatrix}\right\}$ is an orthonormal basis of $V^\perp$.