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Math54 Midterm I, Fall 2016

This is a closed book exam. Everyone is allowed a one-page cheat-sheet but no calculators. You need to justify every one of your answers unless you are asked not to do so. Completely correct answers given without justification will receive little credit. Problems are not necessarily ordered according to difficulties.

Problem	Maximum Score	Your Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total	100	

Your Name: ____________________

Your GSI: ____________________

Your SID: ____________________

1. Solve linear systems of equations $A\,x = b$, where

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

using the row reduction algorithm.

Let $\mathbf{x}$ be $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$. The augmented matrix for $A\mathbf{x} = b$ would be.

$$\left[\begin{array}{ccc:c} 1 & 1 & 1 & 0 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 0 \end{array}\right].$$

① $R_3 \to R_3 - R_1$
② $R_2 \to R_2 - R_1$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 2 & 0 \end{bmatrix}$$

③ $R_1 \to R_1 - R_3$
④ $R_2 \longleftrightarrow R_3$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

⑤ $R_2 \to R_2 - 2R_3$
⑥ $R_1 \to R_1 + R_3$

$$\Rightarrow \left[\begin{array}{ccc:c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

So, this gives $x_1 = 1$, $x_2 = -2$, $x_3 = 1$.

Therefore, $\mathbf{x} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$.

2. Let $\mathbf{T} : \mathcal{R}^3 \to \mathcal{R}^3$ be the linear transformation that reflects each vector through the plane $x_2 = 0$. That is

$$\mathbf{T}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ -x_2 \\ x_3 \end{pmatrix}.$$

Find the standard matrix of $\mathbf{T}$.

The standard matrix of a linear transformation T is

$$\begin{bmatrix} \vdots & \vdots & \vdots \\ T(e_1) & T(e_2) & T(e_3) \\ \vdots & \vdots & \vdots \end{bmatrix}$$

where $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Hence, it is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

3. Mark each statement **True** or **False**. Do not need to justify your answers.

   (a) In order for a matrix B to be the inverse of A, both equations

   $$A\,B = I \quad \text{and} \quad B\,A = I$$

   must be true.

   (b) Each elementary matrix is invertible.

   (c) Let A and $B \in \mathcal{R}^{n\times n}$ be both invertible. Then their product $A\,B$ is also invertible with inverse $A^{-1}\,B^{-1}$.

   (d) If $A \in \mathcal{R}^{n\times n}$ is invertible. Then the equation $A\mathbf{x} = \mathbf{b}$ is consistent for *each* $\mathbf{b} \in \mathcal{R}^n$.

(a) True. (Invertible Matrix Theorem?)

(b) True. (Elementary row operation is reversible!).

(c) False. (AB is invertible, but the inverse is $B^{-1}A^{-1}$ because $(AB)\cdot(B^{-1}A^{-1})$ gives you I_n.).

(d) True. (Since A is invertible, A^{-1} exists so ~~A~~ $x = A^{-1}b$ will be the sol'n for $Ax = b$.)

4. Find a basis for the column space of A, where

$$A = \begin{pmatrix} 0 & 2 & 6 \\ 2 & 2 & 16 \\ -1 & 0 & -5 \end{pmatrix}.$$

A basis for the column space of A can be obtained by finding the pivot columns of A.

$$\begin{bmatrix} 0 & 2 & 6 \\ 2 & 2 & 16 \\ -1 & 0 & -5 \end{bmatrix} \xrightarrow[\text{③ } R_2 \to R_2/2 \quad \text{④ } R_3 \to R_3/2]{\text{① } R_3 \leftrightarrow R_1 \quad \text{② } R_1 \to (-1)R_1} \begin{bmatrix} 1 & 0 & 5 \\ 1 & 1 & 8 \\ 0 & 1 & 3 \end{bmatrix} \xrightarrow[\text{⑥ } R_2 \to R_2 - R_3]{\text{⑤ } R_2 \to R_2 - R_1} \begin{bmatrix} 1 & 0 & 5 \\ 0 & 0 & 0 \\ 0 & 1 & 3 \end{bmatrix}$$

$$\xrightarrow{\text{⑦ } R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}.$$ So, the first and the second column of the original A will be pivot columns.

$$\Rightarrow \left\{ \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} \right\}$$ is a basis for Col A.

5. (a) Use Cramer's Rule to solve

$$A\mathbf{x} = \mathbf{b}, \quad \text{where} \quad A = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

(b) Compute the determinant of

$$A = \begin{pmatrix} 0 & 0 & 6 \\ 0 & 2 & 16 \\ -1 & 0 & -5 \end{pmatrix}.$$

(a) Let $\cancel{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. First of all, we need to check if $\det A \neq 0$. It is.

Cramer's Rule : $x_1 = \dfrac{\det A_1(b)}{\det A} = \dfrac{\det \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}}{\det \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}} = \dfrac{5}{1} = 5$

$x_2 = \dfrac{\det A_2(b)}{\det A} = \dfrac{\det \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}}{\det \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}} = \dfrac{-2}{1} = -2.$

(b) Choose the first column to use cofactor expansion.

$\det A = \cancel{\det} \underbrace{(-1)^{3+1}}_{(-1)^{i+j}} \cdot \underbrace{(-1)}_{a_{ij}} \cdot \det \begin{bmatrix} 0 & 6 \\ 2 & 16 \end{bmatrix}$

$= 1 \cdot (-1) \cdot (-2 \cdot 6) = 12.$