THIS IS NOT THE PERFECT SOLUTION.

Prof. Ming Gu, 861 Evans, tel: 2-3145
Email: mgu@math.berkeley.edu

PLEASE USE THIS AT YOUR OWN RISK!

Math54 Sample Final Exam, Spring 2019

This is a closed everything exam, except a standard one-page cheat sheet (on one-side only). You need to justify every one of your answers. Completely correct answers given without justification will receive little credit. Problems are not necessarily ordered according to difficulties. You need not simplify your answers unless you are specifically asked to do so.

Problem	Maximum Score	Your Score
1	16	ㄱ
2	16	ㄴ
3	16	ㄷ
4	16	ㄹ
5	18	ㅁ
6	18	ㅂ
Total	100	ㅅ

Your Name: Dong Gyu Lim

Your GSI: Dong Gyu Lim

Your SID: ********

1. Let

$$A = \begin{pmatrix} 1 & 5 & -2 & 0 \\ -3 & 1 & 9 & -5 \\ 4 & -8 & -1 & 7 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} -7 \\ 9 \\ 0 \end{pmatrix}.$$

Find all the solutions to the equation

$$A\mathbf{x} = \mathbf{b}.$$

$$\left[\begin{array}{cccc|c} 1 & 5 & -2 & 0 & -7 \\ -3 & 1 & 9 & -5 & 9 \\ 4 & -8 & -1 & 7 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 5 & -2 & 0 & -7 \\ 0 & 16 & 3 & -5 & -12 \\ 0 & -28 & 7 & 7 & 28 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & 5 & -2 & 0 & -7 \\ 0 & 16 & 3 & -5 & -12 \\ 0 & -4 & 1 & 1 & 4 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 5 & -2 & 0 & -7 \\ 0 & 0 & 7 & -1 & 4 \\ 0 & -4 & 1 & 1 & 4 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 1 & -1 & 1 & -3 \\ 0 & -4 & 1 & 1 & 4 \\ 0 & 0 & 7 & -1 & 4 \end{array}\right]$$

Let's first find a particular sol'n. $7X_3 - X_4 = 4$. Just try $X_4 = 3$ and $X_3 = 1$. Then, $-4X_2 + X_3 + X_4 = 4 \Rightarrow X_2 = 0$. Now, $X_1 + X_2 - X_3 + X_4 = -3$ gives you $X_1 = -5$.

Now, we can find homogeneous case sol'n: $\left[\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 0 \\ 0 & -4 & 1 & 1 & 0 \\ 0 & 0 & 7 & -1 & 0 \end{array}\right]$.

To avoid fractions, let $X_4 = 7 \cdot X$ for some $X \in \mathbb{R}$. Then, $7X_3 - X_4 = 0$ gives $X_3 = X$ and then $X_2 = 2X$, $X_1 = -8X$.

Hence, all solutions are of the form:

$$\begin{bmatrix} -5 \\ 0 \\ 1 \\ 3 \end{bmatrix} + \begin{bmatrix} -8 \\ 2 \\ 1 \\ 7 \end{bmatrix} X$$

2. Let V be the vector space $C[-1,1]$, define the inner product

$$\langle f, g\rangle = \int_{-1}^{1} f(x)\, g(x)dx,$$

for any $f, g \in C[-1,1]$. Find an orthogonal basis for the subspace spanned by the polynomials $1, x$ and x^2.

To find an orthogonal basis, we need to apply Gram-Schmidt Orthogonalization Process: $v_1 = 1$, $v_2 = x$, $v_3 = x^2$.

$X_1 := v_1 = 1$

$X_2 := v_2 - \frac{\langle X_1, v_2\rangle}{\langle X_1, X_1\rangle} \cdot X_1 = x - \frac{\langle 1, x\rangle}{\langle 1, 1\rangle} 1$ and $\langle 1, x\rangle = \int_{-1}^{1} 1\cdot x\,dx = 0.$

$= x - \frac{0}{2}\cdot 1$ $\quad \langle 1, 1\rangle = \int_{-1}^{1} 1\cdot 1\,dx = 2.$

$= x.$

$X_3 := v_3 - \frac{\langle X_1, v_3\rangle}{\langle X_1, X_1\rangle} X_1 - \frac{\langle X_2, v_3\rangle}{\langle X_2, X_2\rangle} X_2$ $\quad \langle 1, x^2\rangle = \int_{-1}^{1} 1\cdot x^2 dx = \frac{2}{3}.$

$= x^2 - \frac{\langle 1, x^2\rangle}{\langle 1, 1\rangle} 1 - \frac{\langle x, x^2\rangle}{\langle x, x\rangle} x$ $\quad \langle x, x^2\rangle = \int_{-1}^{1} x\cdot x^2 dx = 0$

($\because x^3$: odd function)

$= x^2 - \frac{2/3}{2}\cdot 1 - \frac{0}{2/3}\cdot x$ $\quad \langle x, x\rangle = \int_{-1}^{1} x\cdot x\,dx = \frac{2}{3}.$

$= x^2 - 1/3.$

So, by Gram-Schmidt, an orthogonal basis for the subspace spanned by the polynomials $1, x$, and x^2 is $\{1, x, x^2 - 1/3\}$.

3. Let

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 4 \end{pmatrix}.$$

(a) Find the eigenvalues and corresponding eigenvectors of A.

(b) Diagonalize A.

The characteristic polynomial is $\det(A-\lambda I) = \det\begin{bmatrix} 1-\lambda & 1 & 2 \\ 1 & 1-\lambda & 2 \\ 2 & 2 & 4-\lambda \end{bmatrix}$

(using 1st column)

$$= (1-\lambda)\big((1-\lambda)(4-\lambda) - 2\cdot 2\big) - \big((4-\lambda) - 2\cdot 2\big) + 2\big(2 - 2(1-\lambda)\big)$$

$$= (1-\lambda)(\lambda^2 - 5\lambda + 4 - 4) - (-\lambda) + 2(2\cdot\lambda)$$

$$= (1-\lambda)(\lambda^2 - 5\lambda) + 5\lambda = -\lambda^3 + \lambda^2 + 5\lambda^2 - 5\lambda + 5\lambda = -\lambda^3 + 6\lambda^2$$

$$= \lambda^2(6-\lambda).$$

Eigenvalues are 0 and 6.

Corresponding eigenvectors are in $\mathrm{Nul}\,A$ and $\mathrm{Nul}(A-6I)$

Nul A part. Row reduction is $\begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ So, we have $x_1 + x_2 + 2x_3 = 0$ and x_2, x_3 can be free $\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_2 - 2x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} x_3.$

So, we have $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$ as a basis of Nul A.

Nul (A-6I) part. $A - 6I = \begin{bmatrix} -5 & 1 & 2 \\ 1 & -5 & 2 \\ 2 & 2 & -2 \end{bmatrix}$. Doing row reduction, one gets $\begin{bmatrix} 1 & 1 & -1 \\ 0 & -6 & 3 \\ 0 & 0 & 0 \end{bmatrix}$.

So, x_3 is free and $x_2 = \frac{1}{2}x_3$, $x_1 = \frac{1}{2}x_3 \Rightarrow \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \right\}$ is a basis.

Diagonalization of A is

$$A = PDP^{-1} \text{ where } P = \begin{bmatrix} 1 & -1 & -2 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \text{ and } D = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

PLEASE READ THE NEXT PAGE.

3. Let

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 4 \end{pmatrix}.$$

(a) Find the eigenvalues and corresponding eigenvectors of A.

(b) Diagonalize A.

In fact, as A is symmetric, one can orthogonally diagonalize A. Then, one can actually avoid the issue of finding P^{-1}!

To do this, we only need to apply Gram-Schmidt to the eigenvectors corresponding to $\lambda=0$. Recall, we have $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$.

Gram-Schmidt does not change the first one and second vector is modifies as

$$\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} - \frac{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}}{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}} \cdot \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} - \frac{2}{2} \cdot \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}.$$

Hence, we have an orthonormal basis consisting of eigenvectors:

$$\left\{ \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}, \begin{bmatrix} -1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix} \right\}.$$

(Orthogonal) Diagonalization of A is

$$\underbrace{\begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{3} & 1/\sqrt{6} \\ 1/\sqrt{2} & -1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & 2/\sqrt{6} \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 6 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{6} & 2/\sqrt{6} \end{bmatrix}}_{P^T (= P^{-1})}.$$

WARNING. There is no easy way to find P^{-1} in the previous page diagonalization.

One might need to use the row reduction OR minor-matrix method.

Some comments about finding eigenvalues without computations.

First observation: A has "essentially" the same row: $[1 \ 1 \ 2]$ and its multiples.

This implies $\det A = 0$ (Invertible Matrix Theorem).

It means $\det(A - 0 \cdot I) = 0$. So, 0 is an eigenvalue.

Furthermore, one can see that $\operatorname{rk} A = 1$ because the row space contains "essentially" one row vector $[1 \ 1 \ 2]$. By rank theorem, $\dim \operatorname{Nul} A = 2$. Therefore, at least two eigenvalues are 0's. Finally, we have the fact that the sum of diagonal entries is the sum of eigenvalues. So, the last eigenvalue should be 6.

$$\underbrace{\text{the sum of diagonal}}_{= 1+1+4} \qquad \underbrace{\text{eigenvalues}}_{= 0+0+?}$$

What is the fact?

Think about 2×2 case:

$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \lambda I\right) = (a-\lambda)(d-\lambda) - bc$$
$$= \lambda^2 - (a+d)\lambda + ad - bc.$$

But, the eigenvalues λ_1 and λ_2 satisfy

$$= (\lambda - \lambda_1)(\lambda - \lambda_2)$$

So, $\underline{\lambda_1 + \lambda_2 = a + d}$ and $\lambda_1 \lambda_2 = ad - bc$.

the sum of eigenvalues. the sum of diagonal entries

4. Let $\mathbf{u} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Find $\mathbf{v} \in \mathbf{R}^3$ such that

$$A = \begin{pmatrix} 1 & -3 & 4 \\ 2 & -6 & 8 \end{pmatrix} = \mathbf{u} \cdot \mathbf{v}^T.$$

Let $v = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$. Then, $u \cdot v^T = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}$

$$= \begin{bmatrix} x_1 & x_2 & x_3 \\ 2x_1 & 2x_2 & 2x_3 \end{bmatrix}.$$

So, $x_1 = 1$, $x_2 = -3$, $x_3 = 4$ and $v = \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix}$

5. Solve the given initial value problem

$$y'' + y' = t + \mathbf{sin}(t), \quad y(0) = 1, \;\; y'(0) = 0.$$

The auxiliary equation is $r^2 + r = 0$. So, we have $r_1 = 0, r_2 = -1$.

We will use the method of undetermined coefficients.

① $y'' + y' = t$: Note that t is $t^1 \cdot e^{0 \cdot t}$ and 0 is a root of the auxiliary equation.

So, we try $y(t) = t^1 (at + b) \cdot e^{0 \cdot t} = at^2 + bt$.

$y'(t) = 2at + b$

$y''(t) = 2a$.

So, $y'' + y' = \underbrace{2at}_{= t} + \underbrace{2a + b}_{= 0}$

$a = \frac{1}{2}$, $b = -1$.

② $y'' + y' = \sin t$: Note that $\sin t$ corresponds to $0 \pm 1 \cdot i$ and this is not a root of the auxiliary equation.

So, we try $y(t) = c \sin t + d \cos t$.

$y'(t) = c \cos t - d \sin t$

$y''(t) = -c \sin t - d \cos t$

So, $y'' + y' = \underbrace{(-c-d)}_{= 1} \sin t + \underbrace{(c-d)}_{= 0} \cos t$

$c = d = -\frac{1}{2}$.

③ Finally, we need homogeneous case solution:

As $r_1 = 0, r_2 = -1$ are roots, we have $\underbrace{e^{0t}}_{= 1}$ and e^{-t}.

Combining ①, ②, ③ one gets $y(t) = c_1 + c_2 \cdot e^{-t} + \frac{1}{2}t^2 - t - \frac{1}{2}\sin t - \frac{1}{2}\cos t$.

Plugging in $t=0$, we have $1 = c_1 + c_2 - \frac{1}{2}$.

" after calculating $y'(t) = -c_2 \cdot e^{-t} + t - 1 - \frac{1}{2}\cos t + \frac{1}{2}\sin t$.

we have $0 = -c_2 - 1 - \frac{1}{2}$. So, $c_1 = 3$, $c_2 = -\frac{3}{2}$.

Answer: $y(t) = 3 - \frac{3}{2}e^{-t} + \frac{1}{2}t^2 - t - \frac{1}{2}\sin t - \frac{1}{2}\cos t$.

6. (a) Find the values of the positive parameter λ for which the given problem below has a nontrivial solution.

$$y'' + \lambda y = 0 \quad \text{for} \quad 0 < x < \pi; \quad y(0) = 0, \;\; y'(\pi) = 0.$$

(b) Compute the Fourier Cosine series of the function $f(x) = x$ on the interval $[0, \pi]$.

(a) The auxiliary equation is $r^2 + \lambda = 0$ and we have $\lambda > 0$ by assumption. So, the roots are $r = \pm\sqrt{\lambda}\, i$. So, the solution is of the form

$$y(t) = C_1 \cos\sqrt{\lambda} t + C_2 \sin\sqrt{\lambda} t.$$

Plugging in $t = 0$, we have $0 = C_1$. So, $y(t) = C_2 \sin\sqrt{\lambda} t$.
Plugging in $t = \pi$ after finding $y'(t) = \sqrt{\lambda}\, C_2 \cos\sqrt{\lambda} t$, we have

$$0 = \sqrt{\lambda}\cdot C_2 \cos\sqrt{\lambda}\pi.$$ If $C_2 = 0$, then $y(t) = 0$ which is the trivial solution!

So, $\cos\sqrt{\lambda}\pi$ should be zero.
Now, recall $\cos x = 0 \iff x = n\pi - \frac{1}{2}\pi$ for some integer n.
$\therefore \sqrt{\lambda} = n - \frac{1}{2}$ for some integer n. So, $\lambda = \frac{1}{4}(2n-1)^2$ for some integer n.
So, possible λ's are $\frac{1}{4}, \frac{9}{4}, \frac{25}{4}, \frac{49}{4}, \ldots$.

(b) We can use the formula: $a_n = \frac{2}{\pi}\int_0^\pi x\cos nx\,dx$ then Fourier cosine series is $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{\pi}$.

For $n = 0$, we have $a_0 = \frac{2}{\pi}\int_0^\pi x\,dx = \frac{2}{\pi}\frac{x^2}{2}\Big|_0^\pi = \pi$.
For $n > 0$, we have

$$a_n = \frac{2}{\pi}\int_0^\pi \underbrace{x}_{u}\, \underbrace{\cos nx}_{v'}\,dx = \frac{2}{\pi}\cdot \underbrace{x}_{u}\cdot\underbrace{\frac{\sin nx}{n}}_{v}\Big|_0^\pi - \frac{2}{\pi}\int_0^\pi \underbrace{1}_{u'}\cdot\underbrace{\frac{\sin nx}{n}}_{v}\,dx \quad \text{(Integration by parts)}$$

$$= \frac{2}{\pi}(0-0) - \frac{2}{\pi}\cdot -\frac{\cos nx}{n^2}\Big|_0^\pi = \frac{2}{\pi}\left[\frac{\cos n\pi}{n^2} - \frac{1}{n^2}\right].$$

Note that $\cos n\pi = (-1)^n$, so we have $a_n = \frac{2}{\pi}\cdot\frac{1}{n^2}\left((-1)^n - 1\right)$.

The Fourier cosine series of $f(x) = x$ on $[0,\pi]$ is

$$\frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\left((-1)^n - 1\right)\cos nx = \frac{\pi}{2} - \frac{4}{\pi}\left(\cos x + \frac{1}{9}\cos 3x + \frac{1}{25}\cos 5x + \frac{1}{49}\cos 7x + \ldots\right)$$