Quiz 9

APRIL 23, 2019

Name (Last, First)

DongGyu Lim

1. (6pts) Solve the following differential equation by variation of parameters:

$$y'' + y = \tan^2 t$$

Hint. $\frac{\sin^2 t}{\cos t} = \frac{1-\cos^2 t}{\cos t} = \sec t - \cos t$ and $\int \sec t = \log(\tan t + \sec t)$.
Solution. First of all, the auxiliay equation is $r^2 + 1 = 0$, so $r = \pi i$ and we have $y_1(t) = \cos t$ and $y_2(t) = \sin t$. For later use, let's compute $W[y_1, y_2] = y_1 y_2' - y_1' y_2$ first. It is $\cos^2 t + \sin^2 t = 1$. Using variation of parameters, one gets

$$\begin{aligned} v_1' y_1 + v_2' y_2 &= 0 \qquad & v_1' &= -y_2 f/aW[y_1, y_2] \\ v_1' y_1' + v_2' y_2' &= f/a \qquad & v_2' &= y_1 f/aW[y_1, y_2] \end{aligned}$$

where $W[y_1, y_2]$ is the Wronskian.

$v_1' = -\sin t \tan^2 t$ so that $v_1 = \int -\sin t \tan^2 t = -\int \frac{\sin^3 t}{\cos^2 t} dt$.

$$\begin{aligned} -\int \frac{\sin^3 t}{\cos^2 t} &= \int \frac{1-\cos^2 t}{\cos^2 t} \cdot -\sin t dt \\ &= \int \frac{1-c^2}{c^2} dc \qquad \text{(Substition } c = \cos t\text{)} \\ &= -\frac{1}{c} - c = -\sec t - \cos t \end{aligned}$$

Now, $v_2 = \int \cos t \tan^2 t = \int \frac{\sin^2 t}{\cos t}$.

$$\begin{aligned} \int \frac{\sin^2 t}{\cos t} &= \int \frac{1-\cos^2 t}{\cos t} dt \\ &= \int \left(\frac{1}{\cos t} - \cos t \right) dt \\ &= \int \sec t - \int \cos t = \log(\tan t + \sec t) - \sin t \end{aligned}$$

Now, the general solution is

$$\begin{aligned} y(t) &= (-\sec t - \cos t)\cos t + (\log(\tan t + \sec t) - \sin t)\sin t + c_1 \cos t + c_2 \sin t \\ &= -2 + \sin t \log(\tan t + \sec t) + c_1 \cos t + c_2 \sin t. \end{aligned}$$

The answer for the other quiz is $y(x) = \frac{1}{4}\cos 2x \log|\cos 2x| + \frac{1}{2} x \sin 2x + c_1 \cos 2x + c_2 \sin 2x$**.**

2. (4pts) Find a fundamental matrix for the system $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$:

$$\mathbf{A} = \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix}.$$

Solution. The characteristic polynomial is $\lambda^2 - 3\lambda + 2$. So, the roots are 1 and 2. The corresponding eigenvectors are $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$. So, $\mathbf{x}_1(t) = e^{\lambda_1 t} v_1 = \begin{bmatrix} e^t \\ -e^t \end{bmatrix}$ and $\mathbf{x}_2(t) = e^{\lambda_2 t} v_2 = \begin{bmatrix} 2e^{2t} \\ -e^{2t} \end{bmatrix}$ form a fundamental solution set and a fundamental matrix is

$$\mathbf{X}(t) = \begin{bmatrix} e^t & 2e^{2t} \\ -e^t & -e^{2t} \end{bmatrix}.$$

The answer for the other quiz is $\mathbf{X}(t) = \begin{bmatrix} e^{2t} & e^{-2t} \\ -e^{2t} & -5e^{-2t} \end{bmatrix}$**.**