Quiz 7

APRIL 9, 2019

Name (Last, First)

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1. (6pts) Find an SVD of the following matrix:

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & -1 \end{bmatrix}.$$

Solution. First of all, we have

$$A^T A = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}.$$

The eigenvalues are $\lambda_1 = 4$ and $\lambda_2 = 2$. So, the singular values are $\sigma_1 = 2$, $\sigma_2 = \sqrt{2}$. The corresponding eigenvectors of length 1 are $v_1 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$ and $v_2 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$. Now, we have

$$\Sigma = \begin{bmatrix} 2 & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix} \qquad \text{and} \qquad V = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}.$$

The columns of U will satisfy

$$u_1 = \frac{Av_1}{\sigma_1} = \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, u_2 = \frac{Av_2}{\sigma_2} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

u_3 should make $\{u_1, u_2, u_3\}$ to be an orthonormal basis. One can find such u_3 by computing the null space of $\begin{bmatrix} u_1^T \\ u_2^T \end{bmatrix}$. Finally, we get

$$U = \begin{bmatrix} 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{bmatrix}$$ and our singular value decomposition would be

$$A = U\Sigma V^T = \begin{bmatrix} 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}.$$

2. (4pts) Find the principal components of the following data:

$$\begin{bmatrix} 3 & 4 & 2 & 1 & 0 \\ 6 & 3 & 1 & 1 & 4 \end{bmatrix}$$

Solution. The mean of column vectors is $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$, so the mean deviation form B is $\begin{bmatrix} 1 & 2 & 0 & -1 & -2 \\ 3 & 0 & -2 & -2 & 1 \end{bmatrix}$. The covariance matrix is

$$S = \frac{1}{N-1} BB^T = \frac{1}{4} \begin{bmatrix} 10 & 3 \\ 3 & 18 \end{bmatrix}.$$

The principal components of the data is the eigenvectors of S. One actually don't need to consider $\frac{1}{4}$ to find out the eigenvectors, that is, we can find eigenvectors of $4S = \begin{bmatrix} 10 & 3 \\ 3 & 18 \end{bmatrix}$. The characteristic polynomial is $(10-\lambda)(18-\lambda) - 3^2 = \lambda^2 - 28\lambda + 171 = (\lambda - 19)(\lambda - 9)$. Now, the eigenvectors or **principal components** would be

$$\frac{1}{\sqrt{10}} \begin{bmatrix} 1 \\ 3 \end{bmatrix} \qquad \text{and} \qquad \frac{1}{\sqrt{10}} \begin{bmatrix} 3 \\ -1 \end{bmatrix}.$$