Quiz 6

APRIL 2, 2019

Name (Last, First)

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1. (8pts) Let A be the symmetric matrix given below:

$$A = \begin{bmatrix} 4 & -1 & -1 \\ -1 & 4 & -1 \\ -1 & -1 & 4 \end{bmatrix}.$$

a. Check if $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T$ is an eigenvector and find the corresponding eigenvalue.

b. It is known that 5 is an eigenvalue of A. Find all eigenvalues and orthogonally diagonalize A.

c. Let $Q(x, y, z) = 4x^2 + 4y^2 + 4z^2 - 2xy - 2yz - 2xz$. Express $Q(x, y, z)$ as the sum of 3 weighted squares. In other words, transform it into another one with no cross-product term.[1]

Solution. a. $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T$ is just the column vector, so you can just check that it is an eigenvector corresponding to the eigenvalue $\lambda_1 = 2$.

b. Let $\lambda_2 = 5$. Then, $A - \lambda_2 I = \begin{bmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{bmatrix}$. As the columns are the same, the dimension of Col$A$ is 1. Using the rank theorem, one can see that the dimension of the null space is 2. So, (including the vector from a.) there would be 3 linearly independent vectors. So, we don't need to check what the characteristic polynomial is to find that

$$E_{\lambda_1} = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\} \quad \text{and} \quad E_{\lambda_2} = \text{Span}\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

As A is symmetric, E_{λ_1} and E_{λ_2} would be ortogonal. So, we now need to get an orthogonal basis of E_{λ_2} applying Gram-Schmidt process. Now, we need to change $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ into $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ MINUS $\text{proj}_{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}\left(\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right)$. So, we have :

$$\begin{bmatrix} 4 & -1 & -1 \\ -1 & 4 & -1 \\ -1 & -1 & 4 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ -1/\sqrt{6} & 2/\sqrt{6} & -1/\sqrt{6} \end{bmatrix}.$$

c. A is the matrix of the quadratic form. So, using spectral decomposition, one gets $\mathbf{x}^T A \mathbf{x} = \mathbf{x}^T(\lambda_1 v_1 v_1^T + \lambda_2 v_2 v_2^T + \lambda_3 v_3 v_3^T)\mathbf{x} = \lambda_1 (v_1^T \mathbf{x})^T (v_1^T \mathbf{x}) + \lambda_2 (v_2^T \mathbf{x})^T (v_2^T \mathbf{x}) + \lambda_3 (v_3^T \mathbf{x})^T (v_3^T \mathbf{x}) = \lambda_1 (v_1^T \mathbf{x})^2 + \lambda_2 (v_2^T \mathbf{x})^2 + \lambda_3 (v_3^T \mathbf{x})^2$. Note here that $v_i^T \mathbf{x}$ is 1×1, so $(v_i^T \mathbf{x})^T = v_i^T \mathbf{x}$. So, it becomes $\frac{2}{3}(x + y + z)^2 + \frac{5}{2}(-x + z)^2 + \frac{5}{6}(-x + 2y - z)^2$.

2. (2pts) Find the maximum value of $Q(\mathbf{x}) = 7x_1^2 + 3x_2^2 - 2x_1 x_2$, subject to the constraint $x_1^2 + x_2^2 = 1$.[2]

Solution. By setting up $x_2' = 2x_2$, we get the standard constrained optimization: finding the maximum of $7x_1^2 + 3x_2'^2 - 2x_1 x_2'$ under the constraint $x_1^2 + x_2'^2 = 1$. The matrix of the quadratic form is

$$\begin{bmatrix} 7 & -1 \\ -1 & 3 \end{bmatrix}.$$

A theorem of Constrained Optimization tells you that the maximum value you can obtain in this situation is λ_{max}, the largest eigenvalue. The characteristic polynomial is $(7 - \lambda)(3 - \lambda) - (-1)^2 = \lambda^2 - 10\lambda + 20$. So, using the formula, we get the larger eigenvalue as $5 + \sqrt{5}$.

[1] For example, $-x^2 + y^2 - z^2 + 2xy + 4xz = (x + y)^2 - 2(x - z)^2 + z^2$. **Hint** : Use the spectral theorem.

[2] You don't need to find the $\mathbf{x}$ that gives you the mamximum $Q(\mathbf{x})$.