Quiz 5

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Name (Last, First)

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1. (8pts) Let A be the matrix given below:

$$A = \begin{bmatrix} 1 & -2 & 2 \\ -1 & 0 & 2 \\ -1 & -2 & 4 \end{bmatrix}.$$

It is known that the characteristic polynomial of A, say $\chi_A(\lambda)$, factors as $(2-\lambda)^2(1-\lambda)$.

a. Find a basis $\mathcal{B}$ of $\mathbb{R}^3$ such that the $\mathcal{B}$-matrix for the transformation $\mathbf{x} \mapsto A\mathbf{x}$ is a diagonal matrix and write down the diagonal matrix.

b. Find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

Solution. As the characteristic polynomial is given, we know that there are two eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 2$.

$E_{\lambda_1} =$ Nul$(A - \lambda_1 I) =$ Nul $\begin{bmatrix} 0 & -2 & 2 \\ -1 & -1 & 2 \\ -1 & -2 & 3 \end{bmatrix}$. We know that the first and second columns are linearly independent. So, the dimension of the null space should be 1 (Rank Theorem). But, $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ works.

$E_{\lambda_2} =$ Nul$(A - \lambda_2 I) =$ Nul $\begin{bmatrix} -1 & -2 & 2 \\ -1 & -2 & 2 \\ -1 & -2 & 2 \end{bmatrix}$. We can observe that the dimension of the column space would be 1. So, the dimension of the null space would be 2. We also know that $2\times$first+$1\times$third=0 and second+third=0. So, we have two vectors, namely $\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

a. $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ and the $\mathcal{B}$-matrix is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

b.

$$P = \begin{bmatrix} 0 & 2 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \quad \& \quad D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

satisfies it.

2. (2pts) Find any $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ orthogonal to $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$. For this $\mathbf{u}$ and $\mathbf{v}$, let $W =$ Span$\{\mathbf{u}, \mathbf{v}\}$. What is $W^\perp$ then?

Solution. Write down the equation: $x_1 + 2x_2 + 3x_3 = 0$ and $3x_1 + 2x_2 + x_3 = 0$. By solving this linear system (probably using row reduction), we get $x_1 = x_3$, $x_2 = -2x_3$, and x_3 can be free. So, one example of such $\mathbf{x}$ would be $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$. Now, $W^\perp$ is actually $\{\mathbf{x} : x_1 = x_3, x_2 = -2x_3, x_3 : \text{free}\} =$ Span $\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}$.

Note. To solve the linear system, you can build up the coefficient matrix as following:

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}$$

and then find the null space of the matrix.