Quiz 4

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Name (Last, First)

Dong Gyu Lim

1. (7pts) Let A be the matrix given below:

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}.$$

a. Find the characteristic equation and eigenvalues.

b. Compute the **eigenspaces** for each eigenvalues.

Solution. $A - \lambda I = \begin{bmatrix} 1-\lambda & -1 & 1 \\ 0 & 2-\lambda & -1 \\ -1 & -1 & 2-\lambda \end{bmatrix}$. Choose the first column to compute the determinant of $A - \lambda I$. We get $(1-\lambda)((2-\lambda)(2-\lambda)-(-1)(-1))+(-1)((-1)(-1)-1(2-\lambda))$. Simplifying it, we get $(1-\lambda)(\lambda^2-4\lambda+3)-(\lambda-1)$ so that $1-\lambda$ is a factor[1]. Now, we have

$$\det(A-\lambda I) = (1-\lambda)(\lambda^2 - 4\lambda + 3 + 1) = (1-\lambda)(2-\lambda)^2.$$

a. The above one is the characteristic equation and eigenvalues are 1 and 2.

b. The eigenspace corresponding to 1 is Nul$(A - 1 \cdot I)$ =Nul $\begin{bmatrix} 0 & -1 & 1 \\ 0 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}$ =Span$\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$.

The eigenspace corresponding to 2 is Nul$(A - 2 \cdot I)$ =Nul $\begin{bmatrix} -1 & -1 & 1 \\ 0 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix}$ =Span$\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\}$.

2. (3pts) Let $\mathcal{B} = \{1+t, 3+2t\}$ and $\mathcal{C} = \{2+t, 5+3t\}$ be bases of $\mathbb{P}_1$.

a. Find the change-of-coordinates matrices $\underset{\mathcal{C}\leftarrow\mathcal{B}}{\mathcal{P}}$ from $\mathcal{B}$ to $\mathcal{C}$ **and** $\underset{\mathcal{B}\leftarrow\mathcal{C}}{\mathcal{P}}$ from $\mathcal{C}$ to $\mathcal{B}$.

b. Compute $\underset{\mathcal{C}\leftarrow\mathcal{B}}{\mathcal{P}} \times \underset{\mathcal{B}\leftarrow\mathcal{C}}{\mathcal{P}}$ and **explain** your result.

Solution. The first and second columns of $\underset{\mathcal{C}\leftarrow\mathcal{B}}{\mathcal{P}}$ are $\mathcal{C}$-coordinate vectors of $\mathbf{b}_1 = 1+t$ and $\mathbf{b}_2 = 3+2t$. Setting up the equation $a(2+t)+b(5+3t) = 1+t$, one can find $a = -2$ and $b = 1$. So, the first column is $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$. In a similar way, one can compute second column as $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

a. So we have $\underset{\mathcal{C}\leftarrow\mathcal{B}}{\mathcal{P}} = \begin{bmatrix} -2 & -1 \\ 1 & 1 \end{bmatrix}$. Again, similarly, one can find $\underset{\mathcal{C}\leftarrow\mathcal{B}}{\mathcal{P}} = \begin{bmatrix} -1 & -1 \\ 1 & 2 \end{bmatrix}$.

b. By just computing, one can get $\underset{\mathcal{C}\leftarrow\mathcal{B}}{\mathcal{P}} \times \underset{\mathcal{B}\leftarrow\mathcal{C}}{\mathcal{P}} = I_2$. The reason for this is because the change-of-coordinates matrix from $\mathcal{C}$ to $\mathcal{C}$ is the identity matrix.

[1] Of course, you can just expand all the terms to get the degree 3 polynomial and then try to plug in some small numbers like $0, \pm 1, \pm 2$ to find roots of the polynomial to find factors.