Quiz 10

APRIL 30, 2019

Name (Last, First)

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1. (7pts) Find a general solution of the system $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{f}(t)$, where

$$\mathbf{A} = \begin{bmatrix} 0 & 2 \\ -1 & 3 \end{bmatrix}, \qquad \mathbf{f}(t) = \begin{bmatrix} e^t \\ -e^t \end{bmatrix}.$$

Solution. We will have 2 different ways to solve the problem. In either cases, we need eigenvalues and eigenvectors of $\mathbf{A}$. So, let's first compute them. The characteristic polynomial of $\mathbf{A}$ is $\lambda^2 - 3\lambda + 2$. So, the roots are $\lambda_1 = 1$ and $\lambda_2 = 2$. Correspnding eigenvectors would be $v_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Let's start:

Undetermined Coefficients

$\mathbf{f}(t)$ is e^t times some vector, so we can try $\mathbf{x}(t) = \mathbf{c}e^t$. However, e^t is $e^{1\cdot t}$ and 1 is an eigenvalue of $\mathbf{A}$. So, we need to try $\mathbf{x}(t) = (\mathbf{a} + \mathbf{b}t)e^t$. (Recall we had t^s term!) Now, plugging this into the equation, we get

$$\mathbf{a}e^t + \mathbf{b}(e^t + te^t) = \mathbf{A}\mathbf{a}e^t + \mathbf{A}\mathbf{b}te^t + \mathbf{f}(t).$$

Dividing every term by e^t, we get

$$\mathbf{a} + \mathbf{b} + \mathbf{b}t = \mathbf{A}\mathbf{a} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \mathbf{A}\mathbf{b}t.$$

Hence, $\mathbf{A}\mathbf{b} = \mathbf{b}$ and $\mathbf{b} - \begin{bmatrix} 1 \\ -1 \end{bmatrix} = (\mathbf{A} - I)\mathbf{a}$ and this implies that $\mathbf{b} = bv_1$ for some $b \in \mathbb{R}$ and then $\mathbf{b} - \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2b-1 \\ b+1 \end{bmatrix}$. Noting that the columns of $\mathbf{A} - I$ are of the form $\begin{bmatrix} x \\ x \end{bmatrix}$, we can conclude that $2b - 1 = b + 1$ so that $b = 2$. Now, $\mathbf{a}$ can be chosen to be $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$. A particular solution would be

$$\begin{bmatrix} -1 \\ 1 \end{bmatrix} e^t + \begin{bmatrix} 4 \\ 2 \end{bmatrix} te^t.$$

Variation of Parameters

Recall this method: $\mathbf{X}(t)$ is a fundamental matrix. This can be obtained by first finding 2 linearly independent solutions. We already have them, namely, $e^{\lambda_1 t}v_1$ and $e^{\lambda_2 t}v_2$. So, $\mathbf{X}(t) = \begin{bmatrix} 2e^t & e^{2t} \\ e^t & e^{2t} \end{bmatrix}$. Trying $\mathbf{x}_p(t) = \mathbf{X}(t)\mathbf{v}(t)$, one gets the formula

$$\begin{aligned} \mathbf{v}(t) &= \int \mathbf{X}^{-1}(t)\mathbf{f}(t)dt \\ &= \int \frac{1}{e^{3t}} \begin{bmatrix} e^{2t} & -e^{2t} \\ -e^t & 2e^t \end{bmatrix} \begin{bmatrix} e^t \\ -e^t \end{bmatrix} dt \\ &= \int \frac{1}{e^{3t}} \begin{bmatrix} 2e^{3t} \\ -3e^{2t} \end{bmatrix} dt \\ &= \int \begin{bmatrix} 2 \\ -3e^{-t} \end{bmatrix} dt = \begin{bmatrix} 2t \\ 3e^{-t} \end{bmatrix}. \end{aligned}$$

Now, a particular solution would be

$$\mathbf{X}(t)\mathbf{v}(t) = \begin{bmatrix} 2e^t & e^{2t} \\ e^t & e^{2t} \end{bmatrix} \begin{bmatrix} 2t \\ 3e^{-t} \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} te^t + \begin{bmatrix} 3 \\ 3 \end{bmatrix} e^t.$$

A general solution is (please check that both give the same thing indeed)

$$\begin{bmatrix} -1 \\ 1 \end{bmatrix} e^t + \begin{bmatrix} 4 \\ 2 \end{bmatrix} te^t + c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^t + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t} \text{ OR } \begin{bmatrix} 3 \\ 3 \end{bmatrix} e^t + \begin{bmatrix} 4 \\ 2 \end{bmatrix} te^t + c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^t + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t}.$$

2. (3pts) Classify the nature of the origin as an attractor, repeller, or saddle point of the dynamical system described by $\mathbf{x}' = A\mathbf{x}$. Find the directions of greatest attraction and/or repulsion.

$$A = \begin{bmatrix} -1 & -2 \\ 1 & -4 \end{bmatrix}$$

Solution. The characteristic polynomial is $\lambda^2 + 5\lambda + 6$. So, the roots are -2 and -3. The corresponding eigenvectors are $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. As the eigenvalues are less than 0, the origin is an **attractor**. The direction of greatest attraction is an eigenvector corresponding to -3 as $-3 < -2$, that is, $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.