Very Naive and Possibly (In)Correct Answers of Final

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1. The vectors

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 2 \\ -8 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} -3 \\ 7 \end{pmatrix}$$

span $\mathbf{R}^2$ but do not form a basis. Find TWO different ways to express $\mathbf{u} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 5 \cdot \begin{pmatrix} 1 \\ -3 \end{pmatrix} + (-2)\begin{pmatrix} 2 \\ -8 \end{pmatrix} + 0 \cdot \begin{pmatrix} -3 \\ 7 \end{pmatrix} \quad \text{OR any } \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix} + \begin{pmatrix} 5 \\ -1 \\ 1 \end{pmatrix} t$$

$$= 0 \cdot \begin{pmatrix} 1 \\ -3 \end{pmatrix} + (-1) \cdot \begin{pmatrix} 2 \\ -8 \end{pmatrix} + (-1) \cdot \begin{pmatrix} -3 \\ 7 \end{pmatrix} \quad \text{satisfies } \begin{pmatrix} 1 \\ 1 \end{pmatrix} = x \cdot \begin{pmatrix} 1 \\ -3 \end{pmatrix} + y \begin{pmatrix} 2 \\ -8 \end{pmatrix} + z \begin{pmatrix} -3 \\ 7 \end{pmatrix}$$

2. • Compute the inverse of the matrix $A = \begin{pmatrix} 1 & 2 \\ 5 & 3 \end{pmatrix}$.

• Compute the determinants of the following matrices

$$B = \begin{pmatrix} 0 & 0 & 4 \\ 0 & 1 & 2 \\ 2 & 2 & 4 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & 2 \\ 2 & 2 & 4 \end{pmatrix}.$$

• $\frac{1}{7}\begin{pmatrix} -3 & 2 \\ 5 & -1 \end{pmatrix}$ OR $\frac{1}{-7}\begin{pmatrix} 3 & -2 \\ -5 & 1 \end{pmatrix}$ • $\det B = -8$ and $\det C = 4$

3. Compute the eigendecomposition and the SVD of the following matrix

$$A = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}.$$

• Diagonalization: $A = PDP^{-1}$ where $P = \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$ and $D = \begin{pmatrix} 1+2i & 0 \\ 0 & 1-2i \end{pmatrix}$

• SVD: $\underbrace{\begin{pmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix}}_{U} \cdot \underbrace{\begin{pmatrix} \sqrt{5} & 0 \\ 0 & \sqrt{5} \end{pmatrix}}_{\Sigma} \cdot \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{V^T}.$

4. • Let $U \in \mathbf{R}^{n \times n}$ be an orthogonal matrix. Show that if $\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ is an orthogonal basis for $\mathbf{R}^n$, then so is $\{U\mathbf{v}_1, U\mathbf{v}_2, \cdots, U\mathbf{v}_n\}$.

• Use the Gram-Schmidt Process to compute a set of orthogonal vectors from the vectors

$$\mathbf{u}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{u}_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{u}_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

• Short proof: recall that $u \cdot v = u^T \cdot v$. Now, if U is an orthogonal matrix, $Uv_i \cdot Uv_j = (Uv_i)^T \cdot (Uv_j) = v_i^T U^T \cdot Uv_j = v_i^T \cdot v_j = v_i \cdot v_j$. Orthogonal basis = Orthogonal set w/ nonzero vectors. As $\|Uv_i\| = \|v_i\|$ by above computation, we have $\|Uv_i\|$'s nonzero and $Uv_i \cdot Uv_j = v_i \cdot v_j = 0$ if $i \neq j$, so orthogonal. $\boxed{\text{Fin}}$

- Ordering u_1, u_2, u_3, one gets $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \frac{1}{4}\begin{pmatrix} 1 \\ 1 \\ 1 \\ -3 \end{pmatrix}, \frac{1}{3}\begin{pmatrix} 1 \\ 1 \\ -2 \\ 0 \end{pmatrix} \right\}$.

  Ordering u_3, u_2, u_1, one gets $\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$.

5. Show that the functions $\left\{\sin\left(\frac{1}{2}x\right), \sin\left(\frac{3}{2}x\right), \cdots, \sin\left(\frac{2n+1}{2}x\right), \cdots\right\}$ are orthogonal under the inner product

$$< f(x), g(x) > \stackrel{def}{=} \int_0^\pi f(x)\, g(x)\, dx.$$

For any $N > 1$, find the orthogonal projection $S_N(x)$ of the function $h(x) = x$ onto the subspace **Span** $\left\{\sin\left(\frac{1}{2}x\right), \sin\left(\frac{3}{2}x\right), \cdots, \sin\left(\frac{2N+1}{2}x\right)\right\}$.
(HINT: You might find this formula below useful: $\sin(\alpha)\sin(\beta) = \frac{1}{2}(\cos(\alpha-\beta) - \cos(\alpha+\beta))$.

- $$\left\langle \sin\frac{2n+1}{2}x, \sin\frac{2m+1}{2}x \right\rangle = \int_0^\pi \sin\frac{2n+1}{2}x \, \sin\frac{2m+1}{2}x \, dx$$
$$= \int_0^\pi \left(\cos(m-n)x - \cos(m+n+1)x\right)/2 \, dx$$

  we need to show that this is 0 if $m \neq n$.

$$= \frac{1}{2}\left[\frac{\sin(m-n)x}{m-n} - \frac{\sin(m+n+1)x}{m+n+1}\right]_0^\pi = 0.$$

  as $\sin k\pi = 0$ if k is an integer. $\square$

- $$\sum_{n=0}^{N} \frac{8 \cdot (-1)^n}{(2n+1)^2 \cdot \pi} \sin\left(\frac{2n+1}{2}\right)x \quad \left(\text{Some computations: } \begin{array}{l} \int_0^\pi x \sin\frac{2n+1}{2}x\,dx = \frac{4\cdot(-1)^n}{(2n+1)^2} \\ \int_0^\pi \sin\frac{2n+1}{2}x \sin\frac{2n+1}{2}x\,dx = \pi/2 \end{array} \right)$$

6. Solve the initial value problem

$$y'' + 2y' + y = t + e^{-t}, \quad y(0) = 1, \quad y'(0) = 0.$$

$$y(t) = \frac{1}{2}t^2 \cdot e^{-t} + t - 2 + 3 \cdot e^{-t} + 2t \cdot e^{-t} \quad \text{OR}$$
$$= t - 2 + \left(\frac{1}{2}t^2 + 2t + 3\right) \cdot e^{-t}.$$