Final Exam in a nutshell

- **time:** Tuesday, May 14, from 7:00-10:00PM, in Pimentel 1.
  - **DSP time:** Tuesday, May 14, from 4:00-10:00PM in 1015 Evans Hall, with DSP letter only.
- **scope:** every section covered in class.
  - Except PCA in Chapter 7, Heat equations in 2 dimensions.
  - 4 Questions from Part I, 2 Questions from Part II.
- **cheat sheet:** one-page/one-sided cheat sheet.
- **background:**
  - Basic knowledge integrating simple functions (exponential, trigonometric functions, polynomials)
  - Basic trigonometric identities.
  - Basic knowledge of complex numbers.
- **OH:** M: 12:15-1:00PM, 4:10-5:30PM, W: 4:10-5:30PM, F: 12:30-2:00PM.

Formula for final grade

- Weekly home work sets;
  Count best 10, total 15 points.
- Weekly Quizzes;
  Count best 10, total 15 points.
- Two midterm exams:
  the worse is 15 points, the better 25 points.
- 1 final exam, 30 points.

Formula for final grade

- Weekly home work sets;
  Count best 10, total 15 points.
- Weekly Quizzes;
  Count best 10, total 15 points.
- Two midterm exams:
  the worse is 15 points, the better 25 points.
- 1 final exam, 30 points.
- **If you miss one midterm, your other midterm and the final will be worth 30 and 40 points, respectively.**

Grade Scale

- **A**- to **A+**: at least 85 points;
- **B**- to **B+**: between 70 and 85 points;
- **C**- to **C+**: between 60 and 70 points;
- **D**: between 55 and 60 points;
- **F**: less than 55 points.

Typically over 2/3 of class receives *B* level grades or above.

How to review

- No need to go beyond the book
- Best to go beyond homework: practice your skills with exercises in the book
- Exam problems are either taken or modified from the book.

Problem distribution

- One question from Chapter 1 through Chapter 3: Linear algebra, Matrix algebra, Determinants
- One question from Chapter 4: Vector Spaces
- One question from Chapter 6: Least Squares
- One question from Chapters 5 and 7: Eigenvalues and SVD
- One question from Chapters 4 and 9: Linear ODEs
- One question from Chapter 10: Fourier Analysis and Heat equation

Sample Problem 1: (Problem 17, p. 90)

Explain why a set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ in $\mathcal{R}^5$ must be linearly independent when $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent and $\mathbf{v}_4 \notin \textbf{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

Sample Problem 1: (Problem 17, p. 90)

Explain why a set $\{\mathbf{v}_1,\, \mathbf{v}_2,\, \mathbf{v}_3,\, \mathbf{v}_4\}$ in $\mathcal{R}^5$ must be linearly independent when $\{\mathbf{v}_1,\, \mathbf{v}_2,\, \mathbf{v}_3\}$ is linearly independent and $\mathbf{v}_4 \notin \mathbf{Span}\,\{\mathbf{v}_1,\, \mathbf{v}_2,\, \mathbf{v}_3\}$.

SOLUTION: We prove the claim by contradiction. Assume that $\{\mathbf{v}_1,\, \mathbf{v}_2,\, \mathbf{v}_3,\, \mathbf{v}_4\}$ in $\mathcal{R}^5$ was linearly dependent. Then there would be constants $c_1, \cdots, c_4$, not all of which were 0, such that

$$c_1\, \mathbf{v}_1 + c_2\, \mathbf{v}_2 + c_3\, \mathbf{v}_3 + c_4\, \mathbf{v}_4 = \mathbf{0}. \quad (\ell)$$

There are two cases to consider in equation (ℓ)

- If $c_4 = 0$, then (ℓ) would become

$$c_1\, \mathbf{v}_1 + c_2\, \mathbf{v}_2 + c_3\, \mathbf{v}_3 = \mathbf{0},$$

  for some constants c_1, c_2, c_3, not all of which were 0, which contradicts the assumption that $\{\mathbf{v}_1,\, \mathbf{v}_2,\, \mathbf{v}_3\}$ is linearly independent.

Sample Problem 1: (Problem 17, p. 90)

Explain why a set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ in $\mathcal{R}^5$ must be linearly independent when $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent and $\mathbf{v}_4 \notin \textbf{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

SOLUTION: We prove the claim by contradiction. Assume that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ in $\mathcal{R}^5$ was linearly dependent. Then there would be constants $c_1, \cdots, c_4$, not all of which were 0, such that

$$c_1\,\mathbf{v}_1 + c_2\,\mathbf{v}_2 + c_3\,\mathbf{v}_3 + c_4\,\mathbf{v}_4 = \mathbf{0}. \quad (\ell)$$

There are two cases to consider in equation (ℓ)

- If $c_4 \neq 0$, then (ℓ) would become

  $$\mathbf{v}_4 = -\frac{1}{c_4}\left(c_1\,\mathbf{v}_1 + c_2\,\mathbf{v}_2 + c_3\,\mathbf{v}_3\right),$$

  which contradicts the assumption that $\mathbf{v}_4 \notin \textbf{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

Thus equation (ℓ) can't be true, meaning $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ must be linearly independent. $\square$

Sample Problem 2: (Problem 5, p. 265)

Consider the polynomials

$$\begin{aligned} \mathbf{p}_1(t) &= 1+t, \quad \mathbf{p}_2(t) = 1-t, \quad \mathbf{p}_3(t) = 4, \\ \mathbf{p}_4(t) &= t+t^2, \quad \mathbf{p}_5(t) = 1+2t+t^2. \end{aligned}$$

Let H be the subspace of $\mathcal{P}_3$ spanned by the set $\mathcal{S} = \{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4, \mathbf{p}_5\}$. Construct a basis for H.

Sample Problem 2: (Problem 5, p. 265)

Consider the polynomials

$$\begin{aligned}\mathbf{p}_1(t) &= 1+t, \quad \mathbf{p}_2(t) = 1-t, \quad \mathbf{p}_3(t) = 4, \\ \mathbf{p}_4(t) &= t+t^2, \quad \mathbf{p}_5(t) = 1+2\,t+t^2.\end{aligned}$$

Let H be the subspace of $\mathcal{P}_3$ spanned by the set $\mathcal{S} = \{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4, \mathbf{p}_5\}$. Construct a basis for H.

SOLUTION: We will construct a basis for H step-by-step:

- Since $\mathbf{p}_1 \neq 0$, it is linearly independent.
- $\mathbf{p}_1$ and $\mathbf{p}_2$ are not multiples of each other, so $\{\mathbf{p}_1, \mathbf{p}_2\}$ is linearly independent.
- $\mathbf{p}_3 = 2\,\mathbf{p}_1 + 2\,\mathbf{p}_2$, so we ignore $\mathbf{p}_3$ for basis construction.
- $\mathbf{p}_4$ is not a linear combination of $\mathbf{p}_1$ and $\mathbf{p}_2$ since neither contains the monomial t^2. so $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_4\}$ is linearly independent.
- $\mathbf{p}_5 = \mathbf{p}_1 + \mathbf{p}_4$, so we ignore $\mathbf{p}_5$ for basis construction.

Thus one basis for H is $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_4\}$. $\square$

Sample Problem 3: (Problem 19, p. 369)

Let $A \in \mathcal{R}^{m \times n}$. Show that $\text{Nul}\, A = \text{Nul}\, A^T A$.

Sample Problem 3: (Problem 19, p. 369)

Let $A \in \mathcal{R}^{m\times n}$. Show that $\text{NUL } A = \text{NUL } A^T A$.
SOLUTION: We will show that $\text{NUL } A \subseteq \text{NUL } A^T A$ and $\text{NUL } A^T A \subseteq \text{NUL } A$ so they must be the same set.

- Let $\mathbf{x} \in \text{NUL } A$ so that $A\mathbf{x} = \mathbf{0}$. This means $A^T A\mathbf{x} = \mathbf{0}$, or that $\mathbf{x} \in \text{NUL } A^T A$. Thus, $\text{NUL } A \subseteq \text{NUL } A^T A$.
- Let $\mathbf{x} \in \text{NUL } A^T A$ so that $A^T A\mathbf{x} = \mathbf{0}$. This means $\mathbf{x}^T A^T A\mathbf{x} = 0$, or that $\|A\mathbf{x}\|^2 = 0$. Therefore $A\mathbf{x} = \mathbf{0}$ and $\mathbf{x} \in \text{NUL } A$. Thus, $\text{NUL } A^T A \subseteq \text{NUL } A \quad \square$.

Sample Problem 4: Compute the SVD

$A = \begin{pmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{pmatrix}$. Eigenvalues of $A^T A$: $\lambda_1 = 360, \lambda_2 = 90, \lambda_3 = 0$,

unit eigenvectors $\mathbf{v}_1 = \frac{1}{3}\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \quad \mathbf{v}_2 = \frac{1}{3}\begin{pmatrix} -2 \\ -1 \\ 2 \end{pmatrix}, \quad \mathbf{v}_3 = \frac{1}{3}\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}.$

$$A\,\mathbf{v}_1 = \sqrt{360}\,\mathbf{u}_1, \quad A\,\mathbf{v}_2 = \sqrt{90}\,\mathbf{u}_2, \quad , A\,\mathbf{v}_3 = \mathbf{0}.$$

$$\text{where} \quad \mathbf{u}_1 = \frac{1}{\sqrt{10}}\begin{pmatrix} 3 \\ 1 \end{pmatrix}, \quad \mathbf{u}_2 = \frac{1}{\sqrt{10}}\begin{pmatrix} 1 \\ -3 \end{pmatrix}.$$

Putting together

$$A\,(\mathbf{v}_1,\, \mathbf{v}_2,\, \mathbf{v}_3) = \left(\sqrt{360}\,\mathbf{u}_1,\, \sqrt{90}\,\mathbf{u}_2,\, \mathbf{0}\right) = (\mathbf{u}_1,\, \mathbf{u}_2)\begin{pmatrix} \sqrt{360} & 0 & 0 \\ 0 & \sqrt{90} & 0 \end{pmatrix}.$$

$$\text{Therefore} \quad A = (\mathbf{u}_1,\, \mathbf{u}_2)\begin{pmatrix} \sqrt{360} & 0 & 0 \\ 0 & \sqrt{90} & 0 \end{pmatrix}(\mathbf{v}_1,\, \mathbf{v}_2,\, \mathbf{v}_3)^T.$$

Sample Problem 5: Problem 7, p. 191

Find a general solution to the ODE:

$$y'' + 4\,y' + 4\,y = e^{-2\,t}\,\mathbf{ln}\,t. \quad (\ell)$$

SOLUTION: The homogeneous equation implied by (ℓ) is

$$y'' + 4\,y' + 4\,y = 0. \quad (\ell_1)$$

The auxiliary equation to (ℓ_1) is $r^2 + 4\,r + 4 = 0$, which has a double root $r = -2$. Therefore two linearly independent solutions to (ℓ_1) are $y_1(t) = e^{-2\,t}$ and $y_2(t) = t\,e^{-2\,t}$. Their Wronskian is

$$\begin{aligned} \mathbf{Wron}[y_1, y_2](t) &= \mathbf{det}\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \\ &= \mathbf{det}\begin{vmatrix} e^{-2\,t} & t\,e^{-2\,t} \\ -2\,e^{-2\,t} & e^{-2\,t} - 2\,t\,e^{-2\,t} \end{vmatrix} = e^{-4\,t}. \end{aligned}$$

Sample Problem 5: Problem 7, p. 191

Find a general solution to the ODE:

$$y'' + 4\,y' + 4\,y = e^{-2\,t}\,\mathbf{ln}\,t. \quad (\ell)$$

SOLUTION: The homogeneous equation implied by (ℓ) has two linearly independent solutions $y_1(t) = e^{-2\,t}$ and $y_2(t) = t\,e^{-2\,t}$. Their Wronskian is

$$\mathbf{Wron}\,[y_1, y_2]\,(t) \quad = \quad e^{-4\,t}.$$

Now we seek a particular solution to (ℓ) in the form

$$y_p(t) = v_1(t)\;y_1(t) + v_2(t)\;y_2(t)\,, \quad \text{where}$$

$$v_1(t) \quad = \quad -\int^t \frac{y_2(\tau)\;e^{-2\,\tau}\,\mathbf{ln}\,\tau\,d\tau}{\mathbf{Wron}\,[y_1, y_2]\,(\tau)} = -\int^t \tau\,\mathbf{ln}\,\tau\,d\tau = c_1 - \frac{t^2}{2}\,\mathbf{ln}\,\frac{t}{\sqrt{e}},$$

$$v_2(t) \quad = \quad \int^t \frac{y_1(\tau)\;e^{-2\,\tau}\,\mathbf{ln}\,\tau\,d\tau}{\mathbf{Wron}\,[y_1, y_2]\,(\tau)} = \int^t \mathbf{ln}\,\tau\,d\tau = c_2 + t\,\mathbf{ln}\,\frac{t}{e}.$$

General solution is $\quad y(t) = (c_1 + c_2\,t)\,e^{-2\,t} + \left(\frac{t^2}{2}\,\mathbf{ln}\,\frac{t}{e\,\sqrt{e}}\right)\,e^{-2\,t}.$

Sample Problem 6

- Compute the Fourier cosine series for the given function $f(x) = \mathbf{sin}(x)$ on $[0, \pi]$.
- Find a formal solution to

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \pi, \quad t > 0.$$

$$u(x,0) = \mathbf{sin}(x) \quad \forall\, x \in [0,\pi]; \quad \frac{\partial u}{\partial x}(0,t) = 0 = \frac{\partial u}{\partial x}(\pi,t) \quad \forall\, t > 0.$$

Sample Problem 6

- Compute the Fourier cosine series for the given function $f(x) = \mathbf{sin}(x)$ on $[0, \pi]$.
- Find a formal solution to

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \pi, \quad t > 0.$$

$$u(x, 0) = \mathbf{sin}(x) \quad \forall\, x \in [0, \pi]; \quad \frac{\partial u}{\partial x}(0, t) = 0 = \frac{\partial u}{\partial x}(\pi, t) \quad \forall\, t > 0.$$

SOLUTION:

- The Fourier cosine series

$$\widetilde{S}(x) = \frac{\widetilde{a}_0}{2} + \sum_{n=1}^{\infty} \widetilde{a}_n \mathbf{cos}(x), \quad \text{where for } n = 0, 1, \cdots,$$

$$\widetilde{a}_n = \frac{2}{\pi} \int_0^{\pi} \mathbf{sin}(x)\, \mathbf{cos}(n\, x)\, dx = \begin{cases} 0, & \text{if } n \text{ is odd,} \\ -\frac{4}{\pi\,(n+1)\,(n-1)}, & \text{if } n \text{ is even,} \end{cases}$$

$$\widetilde{S}(x) = \frac{2}{\pi} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\mathbf{cos}(2\, n\, x)}{(2\, n + 1)\,(2\, n - 1)}.$$

Sample Problem 6

- Compute the Fourier cosine series for the given function $f(x) = \sin(x)$ on $[0, \pi]$.
- Find a formal solution to

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \pi, \quad t > 0. \\ u(x,0) &= \sin(x) \quad \forall\, x \in [0,\pi]; \quad \frac{\partial u}{\partial x}(0,t) = 0 = \frac{\partial u}{\partial x}(\pi,t) \quad \forall\, t > 0. \end{aligned}$$

Sample Problem 6

- Compute the Fourier cosine series for the given function $f(x) = \sin(x)$ on $[0, \pi]$.
- Find a formal solution to

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \pi, \quad t > 0. \\ u(x, 0) &= \sin(x) \quad \forall\, x \in [0, \pi]; \quad \frac{\partial u}{\partial x}(0, t) = 0 = \frac{\partial u}{\partial x}(\pi, t) \quad \forall\, t > 0. \end{aligned}$$

SOLUTION:

- The formal solution to heat equation is

$$u(x, t) \sim \frac{2}{\pi} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cos(2\,n\,x)}{(2\,n + 1)(2\,n - 1)}\, e^{-4\,n^2\,t}.$$