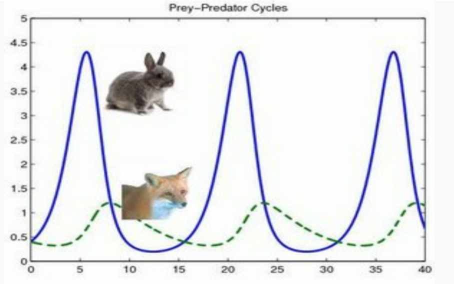
Chap. 9 Matrix Methods for Linear Systems

Predator–Prey equations

$$\frac{dx}{dt} = \alpha x - \beta x y,$$
$$\frac{dy}{dt} = \delta x y - \gamma y.$$

- x, y: prey, predator populations.
- α, β, δ, γ: positive parameters, describing population interactions.



Periodic activity generated by the Predator-Prey model.

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$$\frac{d\,y}{d\,t} = \delta\,x\,y - \gamma\,y.$$

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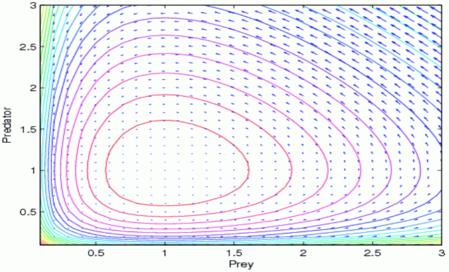


Prey-Predator dynamics as described by the level curves of a conserved quantity. The arrows describe the velocity and direction of solutions. In this simulation, the data

§9.1 Introduction (I)

System of linear equations

$$\begin{aligned} \frac{d\,x}{d\,t} &= -4\,x + 2\,y, \\ \frac{d\,y}{d\,t} &= 4\,x - 2\,y. \end{aligned}$$

In matrix form

$$\mathbf{x}' = A\,\mathbf{x}, \quad \text{where} \quad \mathbf{x} \overset{def}{=} \begin{pmatrix} x \\ y \end{pmatrix}, \; A \overset{def}{=} \begin{pmatrix} -4 & 2 \\ 4 & -2 \end{pmatrix}.$$

§9.1 Introduction (II)

System of linear equations

$$\begin{aligned} x_1' &= 2\,x_1 + t^2\,x_2 + (\cos t)\;x_3, \\ x_2' &= (t + \sin t)\;x_1 + 3\,t\,x_2 + \left(e^t\right)\;x_3, \\ x_3' &= -x_1 + x_2 + x_3. \end{aligned}$$

In matrix form

$$\mathbf{x}' = A\,(t)\;\mathbf{x},$$

where $\quad \mathbf{x} \overset{def}{=} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad A\,(t) \overset{def}{=} \begin{pmatrix} 2 & t^2 & \cos t \\ t + \sin t & 3\,t & e^t \\ -1 & 1 & 1 \end{pmatrix}.$

§9.1 Introduction (III)

$$\underbrace{m}_{\textbf{inertia}} \times y'' \quad + \quad \underbrace{b}_{\textbf{damping}} \times y' \quad + \quad \underbrace{k}_{\textbf{stiffness}} \times y \quad = \quad \mathbf{F_{ext}}\,(t)\,, \qquad (\ell)$$

To turn (ℓ) into system of linear equations, note that $y'' = (y')'$ so that (ℓ) becomes

$$(y')' = -\frac{b}{m}\,(y') - \frac{k}{m}\,y + \frac{\mathbf{F_{ext}}\,(t)}{m}.$$

In matrix form

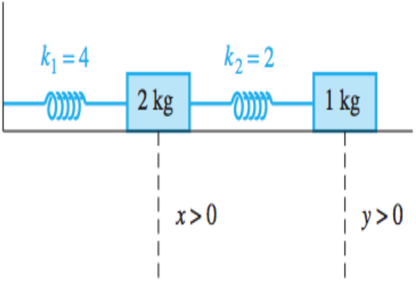
$$\mathbf{x}' = A\,\mathbf{x} + \mathbf{f}\,(t)\,,$$

where $\mathbf{x} \stackrel{def}{=} \begin{pmatrix} y \\ y' \end{pmatrix}, \quad A \stackrel{def}{=} \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{pmatrix}, \quad \mathbf{f}\,(t) = \begin{pmatrix} 0 \\ \frac{\mathbf{F_{ext}}(t)}{m} \end{pmatrix}.$

§9.1 Introduction (IV)

Coupled mass–spring oscillator

$$
\begin{aligned}
2\,\frac{d^2\,x}{d\,t^2} &= -6\,x + 2\,y, \\
\frac{d^2\,y}{d\,t^2} &= 2\,x - 2\,y. \quad (\ell)
\end{aligned}
$$



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$$

To turn system (ℓ) into first order equations, let $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$,

where $\quad x_1 = x,\ x_2 = x_1',\ x_3 = y,\ x_4 = x_3'.$

System (ℓ) becomes $\quad \mathbf{x}' = A\,\mathbf{x}, \quad$ with $\quad A \stackrel{def}{=} \begin{pmatrix} 0 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & -2 & 0 \end{pmatrix}.$

§9.4 Linear systems in normal form

System of n linear differential equations in normal form

$$\mathbf{x}'(t) = A(t)\,\mathbf{x}(t) + \mathbf{f}(t), \quad \mathbf{x}(t), \mathbf{f}(t) \in \mathcal{R}^n, \;\; A(t) \in \mathcal{R}^{n\times n}$$

IVP for system of ODEs: $\quad \mathbf{x}'(t) = A(t)\,\mathbf{x}(t) + \mathbf{f}(t), \quad \mathbf{x}(t_0) = \mathbf{x}_0.$

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Thm. 2, Existence and Uniqueness: Suppose $A(t)$ and $\mathbf{f}(t)$ are continuous on an open interval I that contains t_0. Then, for any initial vector $\mathbf{x}_0$, there exists a unique solution $\mathbf{x}(t)$ on I to IVP.

Linear Dependence of Vector Functions

Definition 1. The m vector functions $\mathbf{x}_1(t), \cdots, \mathbf{x}_m(t) \in \mathcal{R}^n$ are linearly dependent on an interval I if there exist constants $c_1, \cdots, c_m$, not all zero, such that

$$c_1 \mathbf{x}_1(t) + \cdots + c_m \mathbf{x}_m(t) = \mathbf{0}, \quad \text{for all } t \in I.$$

Otherwise, they are linearly independent on I.

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Otherwise, they are linearly independent on I.

Example: Show that $\mathbf{x}_1(t) = \begin{pmatrix} |t| \\ t \\ 1 \end{pmatrix}$, $\mathbf{x}_2(t) = \begin{pmatrix} t \\ |t| \\ 1 \end{pmatrix}$ linearly dependent on $I_1 = [0, 1]$, linearly independent on $I_2 = [-1, 1]$

- On $I_1 = [0, 1]$, $\mathbf{x}_2(t) - \mathbf{x}_1(t) = \mathbf{0}$, thus they are linearly dependent.
- On $I_2 = [-1, 1]$, let $c_1 \mathbf{x}_2(t) + c_2 \mathbf{x}_1(t) = \mathbf{0}$. Then $c_1 |t| + c_2 t = 0$ for $t = \pm 1$. Thus $c_1 = c_2 = 0$.

Definition 2. The **Wronskian** of n vector functions $\mathbf{x}_1(t), \cdots, \mathbf{x}_n(t) \in \mathcal{R}^n$ is

$$\mathbf{W}[\mathbf{x}_1, \cdots, \mathbf{x}_n](t) \stackrel{def}{=} \mathbf{det}(\mathbf{x}_1(t), \cdots, \mathbf{x}_n(t)).$$

Thus, $\mathbf{x}_1(t), \cdots, \mathbf{x}_n(t)$ are linearly independent on interval I if $\mathbf{W}[\mathbf{x}_1, \cdots, \mathbf{x}_n](t) \neq 0$ for any $t \in I$.

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- **Example 1:** $\mathbf{x}_1(t) = \begin{pmatrix} t \\ t^2 \end{pmatrix}$, $\mathbf{x}_2(t) = \begin{pmatrix} t \\ t \end{pmatrix}$ linearly independent since $\mathbf{W}[\mathbf{x}_1, \mathbf{x}_2](t) = t^2 - t^3 \not\equiv 0$.

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- **Example 1:** $\mathbf{x}_1(t) = \begin{pmatrix} t \\ t^2 \end{pmatrix}$, $\mathbf{x}_2(t) = \begin{pmatrix} t \\ t \end{pmatrix}$ linearly independent since $\mathbf{W}[\mathbf{x}_1, \mathbf{x}_2](t) = t^2 - t^3 \not\equiv 0$.

- **Example 2:** $\mathbf{x}_1(t) = \begin{pmatrix} |t| \\ t \end{pmatrix}$, $\mathbf{x}_2(t) = \begin{pmatrix} t \\ |t| \end{pmatrix}$ linearly independent on $I_2 = [-1, 1]$, but $\mathbf{W}[\mathbf{x}_1, \mathbf{x}_2](t) \equiv 0$.

Representation of Solutions

Thm. 3. Let $\mathbf{x}_1(t), \cdots, \mathbf{x}_n(t)$ be n linearly independent solutions to the homogeneous system

$$\mathbf{x}'(t) = A(t)\,\mathbf{x}(t), \quad \mathbf{x}(t) \in \mathcal{R}^n, \;\; A(t) \in \mathcal{R}^{n\times n} \quad (\ell)$$

on interval I, where $A(t)$ is a matrix function continuous on I. Then every solution to (ℓ) on I can be expressed in the form

$$\mathbf{x}(t) = (\mathbf{x}_1(t), \cdots, \mathbf{x}_n(t)) \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix},$$

where $c_1, \cdots, c_n$ are constants.

Example. Verify that

$$\mathbf{x}_1(t) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{2t}, \ \mathbf{x}_2(t) = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} e^{-t}, \ \mathbf{x}_3(t) = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} e^{-t}$$

are fundamental solution set for $\mathbf{x}'(t) = A\mathbf{x}(t), \ A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$

Example. Verify that

$$\mathbf{x}_1(t) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{2t}, \ \mathbf{x}_2(t) = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} e^{-t}, \ \mathbf{x}_3(t) = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} e^{-t}$$

are fundamental solution set for $\mathbf{x}'(t) = A\,\mathbf{x}(t), \quad A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$

SOLUTION:

$$\begin{aligned} \mathbf{x}_1'(t) &= 2\,\mathbf{x}_1(t), & A\,\mathbf{x}_1(t) &= 2\,\mathbf{x}_1(t) \\ \mathbf{x}_2'(t) &= -\mathbf{x}_2(t), & A\,\mathbf{x}_2(t) &= -\mathbf{x}_2(t) \\ \mathbf{x}_3'(t) &= -\mathbf{x}_3(t), & A\,\mathbf{x}_3(t) &= -\mathbf{x}_3(t) \end{aligned}$$

The Wronskian is

$$\begin{aligned} \mathbf{W}[\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3](t) &= \mathbf{det}(\mathbf{x}_1(t), \mathbf{x}_2(t), \mathbf{x}_3(t)) \\ &= \mathbf{det}\begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & -1 \end{pmatrix} = -3. \end{aligned}$$

Representation of Solutions: **Thm. 4.**

- Let $\mathbf{x}_1(t), \cdots, \mathbf{x}_n(t)$ be n linearly independent solutions to

$$\mathbf{x}'(t) = A(t)\,\mathbf{x}(t), \quad \mathbf{x}(t) \in \mathcal{R}^n, \;\; A(t) \in \mathcal{R}^{n\times n} \quad (\ell_1)$$

  on interval I, where $A(t)$ is continuous on I.

- Let $\mathbf{x}_p(t)$ be a particular solution to nonhomogeneous system

$$\mathbf{x}'(t) = A(t)\,\mathbf{x}(t) + \mathbf{f}(t), \quad \mathbf{f}(t) \in \mathcal{R}^n \quad \text{continuous on } I. \quad (\ell_2)$$

Then every solution to (ℓ_2) on I is in the form

$$\mathbf{x}(t) = \mathbf{x}_p(t) + (\mathbf{x}_1(t), \cdots, \mathbf{x}_n(t)) \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix},$$

where $c_1, \cdots, c_n$ are constants.

Homogeneous system, constant coefficients

$$\mathbf{x}'(t) = A\,\mathbf{x}(t), \quad \mathbf{x}(t) \in \mathcal{R}^n, \;\; A \in \mathcal{R}^{n\times n} \quad (\ell)$$

- Assume solution in form $\mathbf{x}(t) = e^{\lambda t}\mathbf{u}$, $\mathbf{u} \neq \mathbf{0}$. Then

$$\mathbf{x}'(t) = \lambda\,\mathbf{u}\,e^{\lambda t}, \quad \text{and} \quad A\,\mathbf{x}(t) = A\,\mathbf{u}\,e^{\lambda t}.$$

- λ, $\mathbf{u}$ must be eigenvalue-eigenvector pair:

$$\Longrightarrow \quad A\,\mathbf{u} = \lambda\,\mathbf{u}, \quad \mathbf{u} \neq \mathbf{0}.$$

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Example. Let $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ in (ℓ). Its eigenvalues are $\lambda_1 = 2, \lambda_2 = -1, \lambda_3 = -1$, with corresponding eigenvectors

$$\mathbf{u}_1(t) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \; \mathbf{u}_2(t) = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \; \mathbf{u}_3(t) = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

Homogeneous system, constant coefficients

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Example. Let $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ in (ℓ). Fundamental solution set

$$\mathbf{x}_1(t) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{2t}, \;\; \mathbf{x}_2(t) = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} e^{-t}, \;\; \mathbf{x}_3(t) = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} e^{-t}.$$

Diagonalizable, homogeneous, constant coef. (SS9.5-6, 5.7)

$$\mathbf{x}'(t) = A\,\mathbf{x}(t), \quad \mathbf{x}(t) \in \mathcal{R}^n, \;\; A \in \mathcal{R}^{n\times n} \quad (\ell_1)$$

- Assume A is diagonalizable:

$$A = U\,\Lambda\,U^{-1}, \quad U = (\mathbf{u}_1, \cdots, \mathbf{u}_n), \; \Lambda = \mathbf{diag}(\lambda_1, \cdots, \lambda_n).$$

  - General solution to (ℓ_1)

$$\mathbf{x}(t) = c_1\, e^{\lambda_1 t}\, \mathbf{u}_1 + \cdots + c_n\, e^{\lambda_n t}\, \mathbf{u}_n. \quad (\ell_2)$$

- If A has n distinct eigenvalues, then A is diagonalizable.
- If A is symmetric, then A is diagonalizable.

Example I: Plot dynamical system trajectory:

$$\mathbf{x}'(t) = A\,\mathbf{x}(t), \quad A = \begin{pmatrix} -1.5 & 0.5 \\ 1 & -1 \end{pmatrix}, \quad \mathbf{x}(0) = \mathbf{x}_0 \quad (\ell)$$

- A is diagonalizable with eigenvalues $\lambda_1 = -0.5, \lambda_2 = -2$:

$$A\,\mathbf{v}_1 = \lambda_1\,\mathbf{v}_1,\ \mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad A\,\mathbf{v}_2 = \lambda_2\,\mathbf{v}_2,\ \mathbf{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

- General solution to (ℓ) : $\mathbf{x}(t) = c_1\,e^{\lambda_1 t}\,\mathbf{v}_1 + c_2\,e^{\lambda_2 t}\,\mathbf{v}_2$.
- c_1, c_2 determined by $\mathbf{x}_0$: $\mathbf{x}(0) = c_1\,\mathbf{v}_1 + c_2\,\mathbf{v}_2 = \mathbf{x}_0$.
- General solution decays to $\mathbf{0}$ because $\lambda_1 < 0, \lambda_2 < 0$.
  ($\mathbf{0}$ is **attractor**)

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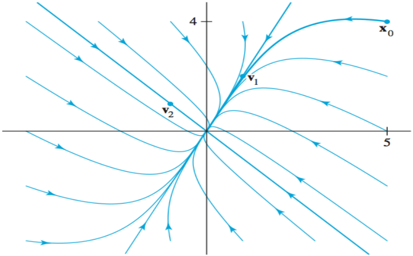

Example II: Plot dynamical system trajectory:

$$\mathbf{x}'(t) = A\,\mathbf{x}(t)\,, \quad A = \begin{pmatrix} 4 & -5 \\ -2 & 1 \end{pmatrix}, \quad \mathbf{x}(0) = \mathbf{x}_0 \quad (\ell)$$

- A is diagonalizable with eigenvalues $\lambda_1 = 6, \lambda_2 = -1$:

$$A\,\mathbf{v}_1 = \lambda_1\,\mathbf{v}_1,\ \mathbf{v}_1 = \begin{pmatrix} -5 \\ 2 \end{pmatrix}, \quad A\,\mathbf{v}_2 = \lambda_2\,\mathbf{v}_2,\ \mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

- General solution to (ℓ) : $\mathbf{x}(t) = c_1\,e^{\lambda_1 t}\,\mathbf{v}_1 + c_2\,e^{\lambda_2 t}\,\mathbf{v}_2$.
- $c_1,\ c_2$ determined by $\mathbf{x}_0$: $\mathbf{x}(0) = c_1\,\mathbf{v}_1 + c_2\,\mathbf{v}_2 = \mathbf{x}_0$.
- $\mathbf{0}$ is **saddle** because $\lambda_1 > 0, \lambda_2 < 0$.

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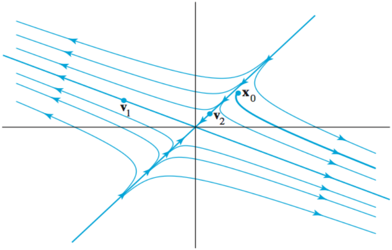

FIGURE 3 The origin as a saddle point.

§9.6 Complex eigenvalues (I)

$$\mathbf{x}'(t) = A\,\mathbf{x}(t), \quad \mathbf{x}(t) \in \mathcal{R}^n, \;\; A \in \mathcal{R}^{n\times n} \quad (\ell)$$

- Assume solution in form $\mathbf{x}(t) = e^{\lambda\, t}\mathbf{u}$, $\mathbf{u} \neq \mathbf{0}$. Then λ, $\mathbf{u}$ must be eigenvalue-eigenvector pair:

$$\Longrightarrow \quad A\,\mathbf{u} = \lambda\,\mathbf{u}, \quad \mathbf{u} \neq \mathbf{0}.$$

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$$\Longrightarrow \quad A\,\mathbf{u} = \lambda\,\mathbf{u}, \quad \mathbf{u} \neq \mathbf{0}.$$

- If $\lambda = \alpha + i\,\beta$ is complex, with $i^2 = -1$ and $\beta \neq 0$, then $\mathbf{u} = \mathbf{a} + i\,\mathbf{b}$ must be complex, with $\mathbf{b} \neq \mathbf{0}$.
- Since $(-i)^2 = -1$, another eigenvalue must be $\overline{\lambda} = \alpha - i\,\beta$, and another eigenvector $\overline{\mathbf{u}} = \mathbf{a} - i\,\mathbf{b}$
- $\mathbf{x}(t) = e^{\lambda\, t}\mathbf{u}$ and $\overline{\mathbf{x}}(t) = e^{\overline{\lambda}\, t}\overline{\mathbf{u}}$ must both be solutions to (ℓ).

§9.6 Complex eigenvalues (II)

$$\mathbf{x}'(t) = A\,\mathbf{x}(t), \quad \mathbf{x}(t) \in \mathcal{R}^n, \;\; A \in \mathcal{R}^{n\times n} \quad (\ell)$$

- $\mathbf{x}(t) = e^{\lambda t}\mathbf{u}$ is solution to (ℓ) with $\lambda = \alpha + i\,\beta$, $\mathbf{u} = \mathbf{a} + i\,\mathbf{b}$.

$$\begin{aligned}
\mathbf{x}(t) &= e^{(\alpha+i\beta)t}\,\mathbf{u} = e^{\alpha t}\left(\cos(\beta t) + i\sin(\beta t)\right)(\mathbf{a} + i\,\mathbf{b}) \\
&= e^{\alpha t}(\cos(\beta t)\,\mathbf{a} - \sin(\beta t)\,\mathbf{b}) + i\,e^{\alpha t}(\sin(\beta t)\,\mathbf{a} + \cos(\beta t)\,\mathbf{b}) \\
&\overset{def}{=} \overbrace{\mathbf{w}_1(t)} \qquad + \qquad i \qquad \overbrace{\mathbf{w}_2(t)}
\end{aligned}$$

- Equation (ℓ) becomes

$$\mathbf{w}_1'(t) + i\,\mathbf{w}_2'(t) = A\,\mathbf{w}_1(t) + i\,A\,\mathbf{w}_2(t),$$

- which is <u>two</u> solutions to (ℓ),

$$\mathbf{w}_1'(t) = A\,\mathbf{w}_1(t), \quad \mathbf{w}_2'(t) = A\,\mathbf{w}_2(t).$$

Complex eigenvalues: **Example I**

Find general solution to

$$\mathbf{x}'(t) = A\,\mathbf{x}(t), \quad \mathbf{x}(t) \in \mathcal{R}^2, \quad A = \begin{pmatrix} -1 & 2 \\ -1 & -3 \end{pmatrix} \quad (\ell)$$

SOLUTION: eigenvalues of $A = -2 \pm i$,

$$A\left(\begin{pmatrix} 2 \\ -1 \end{pmatrix} + i\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = (-2+i)\left(\begin{pmatrix} 2 \\ -1 \end{pmatrix} + i\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right).$$

Therefore

$$\mathbf{w}_1(t) = e^{-2t}\left(\cos(t)\begin{pmatrix} 2 \\ -1 \end{pmatrix} - \sin(t)\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right),$$

$$\mathbf{w}_2(t) = e^{-2t}\left(\sin(t)\begin{pmatrix} 2 \\ -1 \end{pmatrix} + \cos(t)\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right).$$

- general solution is

$$\mathbf{y}(t) = c_1\,\mathbf{w}_1(t) + c_2\,\mathbf{w}_2(t)$$

$$= e^{-2t}\left(\cos(t)\begin{pmatrix} 2c_1 \\ c_2 - c_1 \end{pmatrix} + \sin(t)\begin{pmatrix} 2c_2 \\ -c_2 - c_1 \end{pmatrix}\right).$$

Coupled mass–spring oscillator with fixed ends (I)

$$\begin{aligned} m_1 \frac{d^2 x_1}{d\, t^2} &= -k_1 x_1 + k_2 (x_2 - x_1), \\ m_2 \frac{d^2 x_2}{d\, t^2} &= -k_2 (x_2 - x_1) - k_3 x_2, \end{aligned}$$

where x_1, x_2 are displacements of masses m_1, m_2.

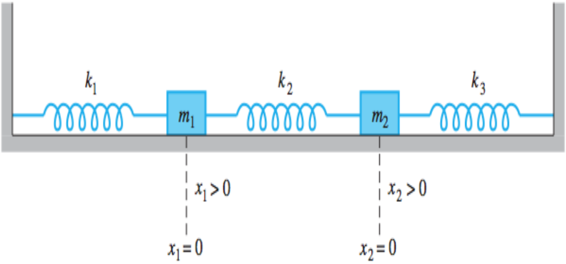

Figure 9.5 Coupled mass–spring system with fixed ends

Coupled mass–spring oscillator with fixed ends (II)

$$
\begin{aligned}
m_1 \frac{d^2 x_1}{d\, t^2} &= -k_1\, x_1 + k_2\, (x_2 - x_1)\,, \\
m_2 \frac{d^2 x_2}{d\, t^2} &= -k_2\, (x_2 - x_1) - k_3\, x_2,
\end{aligned}
$$

where x_1, x_2 are displacements of masses m_1, m_2. Normal form

$$
\mathbf{y}'(t) = A\,\mathbf{y}(t)\,, \quad \text{with} \quad \mathbf{y} = \begin{pmatrix} x_1 \\ x_1' \\ x_2 \\ x_2' \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\frac{k_1+k_2}{m_1} & 0 & \frac{k_2}{m_1} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k_2}{m_2} & 0 & -\frac{k_2+k_3}{m_2} & 0 \end{pmatrix}.
$$

A only has imaginary eigenvalues $\pm\, i\, \beta_1$, $\pm\, i\, \beta_2$.

Coupled mass–spring oscillator with fixed ends (III)

$$\mathbf{y}'(t) = A\,\mathbf{y}(t), \quad \text{with} \quad \mathbf{y} = \begin{pmatrix} x_1 \\ x_1' \\ x_2 \\ x_2' \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\frac{k_1+k_2}{m_1} & 0 & \frac{k_2}{m_1} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k_2}{m_2} & 0 & -\frac{k_2+k_3}{m_2} & 0 \end{pmatrix}$$

A has imaginary eigenvalues $\pm i\,\beta_1$, $\pm i\,\beta_2$. Compute β_1, β_2 for

$$m_1 = m_2 = 1\text{kg}, \; k_1 = 1\text{kg/sec}^2, \; k_2 = 2\text{kg/sec}^2, \; k_3 = 3\text{kg/sec}^2.$$

SOLUTION:

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -3 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & -5 & 0 \end{pmatrix}, \quad \mathbf{det}\,(A - \lambda I) = \lambda^4 + 8\,\lambda^2 + 11 = 0.$$

Thus, $\lambda^2 = -4 \pm \sqrt{5}$.

$$\beta_1 = \sqrt{4 - \sqrt{5}}\,/\text{sec}\,, \quad \beta_2 = \sqrt{4 + \sqrt{5}}\,/\text{sec}\,.$$

Example III: Plot dynamical system trajectory:

$$\mathbf{x}'(t) = A\,\mathbf{x}(t), \quad A = \begin{pmatrix} -2 & -2.5 \\ 10 & -2 \end{pmatrix}, \quad \mathbf{x}(0) = \mathbf{x}_0 \quad (\ell)$$

- A is diagonalizable with complex eigenvalues $\lambda = -2 \pm 5\,i$ and eigenvectors $\begin{pmatrix} \pm\, i \\ 2 \end{pmatrix}$. General solution to (ℓ):

$$\mathbf{x}(t) = c_1 \begin{pmatrix} -\sin(5\,t) \\ 2\cos(5\,t) \end{pmatrix} e^{-2\,t} + c_2 \begin{pmatrix} \cos(5\,t) \\ 2\sin(5\,t) \end{pmatrix} e^{-2\,t}.$$

- c_1, c_2 determined by $\mathbf{x}_0$: $\mathbf{x}(0) = c_1 \begin{pmatrix} 0 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \mathbf{x}_0$.
- $\mathbf{0}$ is **spiral point** because of factor $e^{-2\,t}$.

Example III: Plot dynamical system trajectory:

$$\mathbf{x}'(t) = A\,\mathbf{x}(t), \quad A = \begin{pmatrix} -2 & -2.5 \\ 10 & -2 \end{pmatrix}, \quad \mathbf{x}(0) = \mathbf{x}_0 \quad (\ell)$$

- A is diagonalizable with complex eigenvalues $\lambda = -2 \pm 5\,i$ and eigenvectors $\begin{pmatrix} \pm i \\ 2 \end{pmatrix}$. General solution to (ℓ):

$$\mathbf{x}(t) = c_1 \begin{pmatrix} -\sin(5\,t) \\ 2\cos(5\,t) \end{pmatrix} e^{-2\,t} + c_2 \begin{pmatrix} \cos(5\,t) \\ 2\sin(5\,t) \end{pmatrix} e^{-2\,t}.$$

- $c_1,\ c_2$ determined by $\mathbf{x}_0$: $\mathbf{x}(0) = c_1 \begin{pmatrix} 0 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \mathbf{x}_0$.
- **0** is **spiral point** because of factor $e^{-2\,t}$.

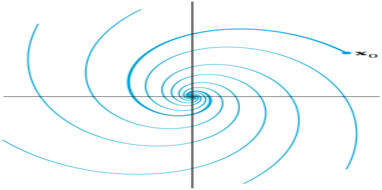

FIGURE 5
The origin as a spiral point.

§9.7 Representation of Solutions: **Thm. 4.**

- Let $\mathbf{x}_1(t), \cdots, \mathbf{x}_n(t)$ be n linearly independent solutions to

$$\mathbf{x}'(t) = A\mathbf{x}(t), \quad \mathbf{x}(t) \in \mathcal{R}^n, \;\; A \in \mathcal{R}^{n\times n}. \quad (\ell_1)$$

- Let $\mathbf{x}_p(t)$ be a particular solution to nonhomogeneous system

$$\mathbf{x}'(t) = A\mathbf{x}(t) + \mathbf{f}(t), \quad \mathbf{f}(t) \in \mathcal{R}^n \quad \text{continuous.} \quad (\ell_2)$$

Then every solution to (ℓ_2) is in the form

$$\mathbf{x}(t) = \mathbf{x}_p(t) + (\mathbf{x}_1(t), \cdots, \mathbf{x}_n(t)) \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix},$$

where $c_1, \cdots, c_n$ are constants.

Example I: Find general solution to

$$\mathbf{x}'(t) = A\,\mathbf{x}(t) + t\,\mathbf{g}, \quad \text{with} \quad A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \quad \mathbf{g} = \begin{pmatrix} -1 \\ 0 \\ -2 \end{pmatrix}. \quad (\ell)$$

SOLUTION:

- Fundamental solution set for $\mathbf{x}'(t) = A\,\mathbf{x}(t)$ is

$$\mathbf{x}_1(t) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{2t}, \quad \mathbf{x}_2(t) = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} e^{-t}, \quad \mathbf{x}_3(t) = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} e^{-t}.$$

- Seek particular solution $\mathbf{x}_p(t) = t\,\mathbf{a} + \mathbf{b}$. (ℓ) becomes

$$\mathbf{a} = A\,(t\,\mathbf{a} + \mathbf{b}) + t\,\mathbf{g}.$$

Therefore $\mathbf{a} = -A^{-1}\,\mathbf{g} = \frac{1}{2}\begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}, \quad \mathbf{b} = A^{-1}\,\mathbf{a} = \frac{1}{4}\begin{pmatrix} 1 \\ -3 \\ 5 \end{pmatrix}.$

Example I: Find general solution $\cdots$

SOLUTION:

- Fundamental solution set for $\mathbf{x}'(t) = A\,\mathbf{x}(t)$ is

$$\mathbf{x}_1(t) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{2t}, \ \mathbf{x}_2(t) = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} e^{-t}, \ \mathbf{x}_3(t) = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} e^{-t}.$$

- Particular solution $\mathbf{x}_p(t) = \frac{t}{2} \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} + \frac{1}{4} \begin{pmatrix} 1 \\ -3 \\ 5 \end{pmatrix}$.
- general solution to (ℓ)

$$\begin{aligned} \mathbf{x}(t) &= \frac{t}{2} \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} + \frac{1}{4} \begin{pmatrix} 1 \\ -3 \\ 5 \end{pmatrix} \\ &\quad + c_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} e^{-t} + c_3 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} e^{-t} \end{aligned}$$

Variation of parameters (I)

- Let $\mathbf{x}_1(t), \cdots, \mathbf{x}_n(t)$ be n linearly independent solutions to
$$\mathbf{x}'(t) = A(t)\,\mathbf{x}(t), \quad \mathbf{x}(t) \in \mathcal{R}^n, \;\; A(t) \in \mathcal{R}^{n\times n}. \quad (\ell_1)$$
- Seek particular solution $\mathbf{x}_p(t) = (\mathbf{x}_1(t), \cdots, \mathbf{x}_n(t)) \cdot \mathbf{v}(t)$ to nonhomogeneous system
$$\mathbf{x}'(t) = A(t)\,\mathbf{x}(t) + \mathbf{f}(t), \quad \mathbf{f}(t) \in \mathcal{R}^n \quad \text{continuous.} \quad (\ell_2)$$
- Equation (ℓ_2) becomes

$$\mathbf{x}_p'(t) = \left(\mathbf{x}_1'(t), \cdots, \mathbf{x}_n'(t) \right) \cdot \mathbf{v}(t) + (\mathbf{x}_1(t), \cdots, \mathbf{x}_n(t)) \cdot \mathbf{v}'(t)$$
$$= \left(\overbrace{A(t)\,\mathbf{x}_1(t)}, \cdots, \overbrace{A(t)\,\mathbf{x}_n(t)} \right) \cdot \mathbf{v}(t) + \mathbf{f}(t).$$

- Therefore $(\mathbf{x}_1(t), \cdots, \mathbf{x}_n(t)) \cdot \mathbf{v}'(t) = \mathbf{f}(t)$.

$$\mathbf{v}(t) = \int^t (\mathbf{x}_1(\tau), \cdots, \mathbf{x}_n(\tau))^{-1} \mathbf{f}(\tau)\, d\tau,$$
$$\mathbf{x}_p(t) = (\mathbf{x}_1(t), \cdots, \mathbf{x}_n(t)) \int^t (\mathbf{x}_1(\tau), \cdots, \mathbf{x}_n(\tau))^{-1} \mathbf{f}(\tau)\, d\tau.$$

Variation of parameters (II)

Find the solution to IVP

$$\mathbf{x}'(t) = A(t)\,\mathbf{x}(t) + \mathbf{f}(t), \quad \mathbf{f}(t) \in \mathcal{R}^n, \quad \mathbf{x}(t_0) = \mathbf{x}_0. \quad (\ell_1)$$

SOLUTION: Let $\mathbf{x}_1(t), \cdots, \mathbf{x}_n(t)$ be n linearly independent solutions to

$$\mathbf{x}'(t) = A(t)\,\mathbf{x}(t), \quad \mathbf{x}(t) \in \mathcal{R}^n, \;\; A(t) \in \mathcal{R}^{n\times n}. \quad (\ell_2)$$

Solution to (ℓ_1) has form

$$\begin{aligned}
\mathbf{x}(t) &= (\mathbf{x}_1(t), \cdots, \mathbf{x}_n(t))\,\mathbf{c} \\
&\quad + (\mathbf{x}_1(t), \cdots, \mathbf{x}_n(t)) \int_{t_0}^{t} (\mathbf{x}_1(\tau), \cdots, \mathbf{x}_n(\tau))^{-1}\,\mathbf{f}(\tau)\; d\tau. \\
\mathbf{x}(t_0) &= (\mathbf{x}_1(t_0), \cdots, \mathbf{x}_n(t_0))\,\mathbf{c}, \quad \Rightarrow \mathbf{c} = (\mathbf{x}_1(t_0), \cdots, \mathbf{x}_n(t_0))^{-1}\,\mathbf{x}(t_0)
\end{aligned}$$

Variation of parameters (III)

Find the solution to IVP

$$\mathbf{x}'(t) = A(t)\,\mathbf{x}(t) + \mathbf{f}(t), \quad \mathbf{f}(t) \in \mathcal{R}^n, \quad \mathbf{x}(t_0) = \mathbf{x}_0. \quad (\ell_1)$$

SOLUTION: Let $\mathbf{x}_1(t), \cdots, \mathbf{x}_n(t)$ be n linearly independent solutions to

$$\mathbf{x}'(t) = A(t)\,\mathbf{x}(t), \quad \mathbf{x}(t) \in \mathcal{R}^n, \;\; A(t) \in \mathcal{R}^{n\times n}. \quad (\ell_2)$$

Solution to (ℓ_1) has form

$$\begin{aligned}\mathbf{x}(t) &= (\mathbf{x}_1(t), \cdots, \mathbf{x}_n(t))\left((\mathbf{x}_1(t_0), \cdots, \mathbf{x}_n(t_0))^{-1}\,\mathbf{x}(t_0)\right) \\ &\quad + (\mathbf{x}_1(t), \cdots, \mathbf{x}_n(t))\int_{t_0}^{t} (\mathbf{x}_1(\tau), \cdots, \mathbf{x}_n(\tau))^{-1}\,\mathbf{f}(\tau)\,d\tau.\end{aligned}$$

Variation of parameters, **Example**

Solve $\mathbf{x}'(t) = \begin{pmatrix} 2 & -3 \\ 1 & -2 \end{pmatrix} \mathbf{x}(t) + \begin{pmatrix} e^{2t} \\ 1 \end{pmatrix}, \quad \mathbf{x}(0) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}. \quad (\ell)$

Variation of parameters, **Example**

Solve $\mathbf{x}'(t) = \begin{pmatrix} 2 & -3 \\ 1 & -2 \end{pmatrix} \mathbf{x}(t) + \begin{pmatrix} e^{2t} \\ 1 \end{pmatrix}, \quad \mathbf{x}(0) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}. \quad (\ell)$

SOLUTION: $\mathbf{x}_1(t) = \begin{pmatrix} 3e^t \\ e^t \end{pmatrix}$ and $\mathbf{x}_2(t) = \begin{pmatrix} e^{-t} \\ e^{-t} \end{pmatrix}$ linearly independent solutions to $\mathbf{x}'(t) = \begin{pmatrix} 2 & -3 \\ 1 & -2 \end{pmatrix} \mathbf{x}(t)$.

$$
\begin{aligned}
\text{Solution } \mathbf{x}(t) &= (\mathbf{x}_1(t), \mathbf{x}_2(t)) \left((\mathbf{x}_1(0), \mathbf{x}_2(0))^{-1} \mathbf{x}(0) \right) \\
&\quad + (\mathbf{x}_1(t), \mathbf{x}_2(t)) \int_0^t (\mathbf{x}_1(\tau), \mathbf{x}_2(\tau))^{-1} \mathbf{f}(\tau) \, d\tau \\
&= \begin{pmatrix} 3e^t & e^{-t} \\ e^t & e^{-t} \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} -1 \\ 0 \end{pmatrix} \\
&\quad + \begin{pmatrix} 3e^t & e^{-t} \\ e^t & e^{-t} \end{pmatrix} \int_0^t \begin{pmatrix} 3e^\tau & e^{-\tau} \\ e^\tau & e^{-\tau} \end{pmatrix}^{-1} \begin{pmatrix} e^{2\tau} \\ 1 \end{pmatrix} d\tau
\end{aligned}
$$

Variation of parameters, **Example**

$$\text{Solution } \mathbf{x}(t) = \begin{pmatrix} 3\,e^{t} & e^{-t} \\ e^{t} & e^{-t} \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} -1 \\ 0 \end{pmatrix} + \begin{pmatrix} 3\,e^{t} & e^{-t} \\ e^{t} & e^{-t} \end{pmatrix} \int_0^t \begin{pmatrix} 3\,e^{\tau} & e^{-\tau} \\ e^{\tau} & e^{-\tau} \end{pmatrix}^{-1} \begin{pmatrix} e^{2\,\tau} \\ 1 \end{pmatrix} d\,\tau$$

Variation of parameters, **Example**

Solution $\mathbf{x}(t)$

$$
\begin{aligned}
\mathbf{x}(t) &= \begin{pmatrix} 3\,e^{t} & e^{-t} \\ e^{t} & e^{-t} \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} -1 \\ 0 \end{pmatrix} \\
&\quad + \begin{pmatrix} 3\,e^{t} & e^{-t} \\ e^{t} & e^{-t} \end{pmatrix} \int_0^t \begin{pmatrix} 3\,e^{\tau} & e^{-\tau} \\ e^{\tau} & e^{-\tau} \end{pmatrix}^{-1} \begin{pmatrix} e^{2\,\tau} \\ 1 \end{pmatrix} d\,\tau \\
&= \frac{1}{2} \begin{pmatrix} 3\,e^{t} & e^{-t} \\ e^{t} & e^{-t} \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \\
&\quad + \frac{1}{2} \begin{pmatrix} 3\,e^{t} & e^{-t} \\ e^{t} & e^{-t} \end{pmatrix} \int_0^t \begin{pmatrix} e^{\tau} - e^{-\tau} \\ -e^{3\,\tau} + 3\,e^{\tau} \end{pmatrix} d\,\tau \\
&= \frac{1}{2} \begin{pmatrix} 3\,e^{t} & e^{-t} \\ e^{t} & e^{-t} \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \\
&\quad + \frac{1}{2} \begin{pmatrix} 3\,e^{t} & e^{-t} \\ e^{t} & e^{-t} \end{pmatrix} \begin{pmatrix} e^{t} + e^{-t} - 2 \\ -\frac{1}{3}e^{3\,t} + 3\,e^{t} - \frac{8}{3} \end{pmatrix} \\
&= \frac{1}{6} \begin{pmatrix} -27\,e^{t} - 5\,e^{-t} + 8\,e^{2\,t} + 18 \\ -9\,e^{t} - 5\,e^{-t} + 2\,e^{2\,t} + 12 \end{pmatrix}
\end{aligned}
$$