§4.1 Damped mass–spring oscillator (I)

- mass m attached to a spring fixed at one end
- mass m moves due to external force $\mathbf{F}_{\mathbf{ext}}(t)$
- mass m is slowed down by SPRING and FRICTION.
- Goal: derive a differential equation to describe the motion

$y = y(t)$, time-dependent displacement from EQUILIBRIUM POINT.

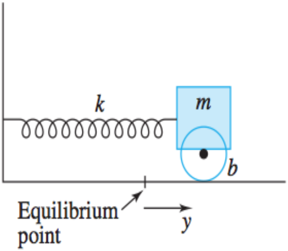


Figure 4.1 Damped mass–spring oscillator

§4.1 Damped mass–spring oscillator (II)

Newton's second law– total force = mass m times acceleration

- total force:
  - Spring resistance: $\mathbf{F}_{\mathbf{spring}} \stackrel{def}{=} -k\,y, \quad k = \textit{stiffness}.$
  - Friction:

$$\mathbf{F}_{\mathbf{friction}} \stackrel{def}{=} -b\,\frac{d\,y}{d\,t} = -b\,y', \quad b = \textit{damping coefficient}$$

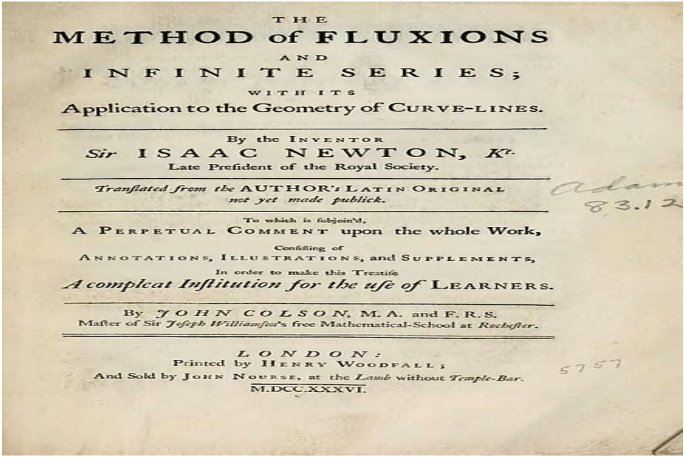
  - total force $= \mathbf{F}_{\mathbf{ext}}(t) - k\,y - b\,y'$
- mass m times acceleration $= m\,\frac{d^2\,y}{d\,t^2} = m\,y''$.

Differential equation according to Newton's second law

$$m\,y'' = \mathbf{F}_{\mathbf{ext}}(t) - k\,y - b\,y', \quad \text{or}$$

$$\underbrace{m}_{\textbf{inertia}} \times y'' + \underbrace{b}_{\textbf{damping}} \times y' + \underbrace{k}_{\textbf{stiffness}} \times y = \mathbf{F}_{\mathbf{ext}}(t),$$

World's first Differential equations in 1671


THE
METHOD of FLUXIONS
AND
INFINITE SERIES;
WITH ITS
Application to the Geometry of CURVE-LINES.

By the INVENTOR
Sir ISAAC NEWTON, *Kt.*
Late Prefident of the Royal Society.

Tranflated from the AUTHOR's LATIN ORIGINAL
not yet made publick.

To which is fubjoin'd,
A PERPETUAL COMMENT upon the whole Work,
Confifting of
ANNOTATIONS, ILLUSTRATIONS, and SUPPLEMENTS,
In order to make this Treatife
A compleat Inftitution for the ufe of LEARNERS.

By *JOHN COLSON*, M.A. and F.R.S.
Mafter of Sir *Jofeph Williamfon*'s free Mathematical-School at *Rochefter*.

LONDON:
Printed by HENRY WOODFALL;
And Sold by JOHN NOURSE, at the *Lamb* without *Temple-Bar*.
M.DCC.XXXVI.

§4.2 Homogeneous linear equations

- Homogeneous linear 2^{nd} order constant-coefficient differential equation $(a \neq 0)$

$$a\,y'' + b\,y' + c\,y = 0, \quad (\ell_1),$$

- General linear 2^{nd} order constant-coefficient DE $(a \neq 0)$

$$a\,y'' + b\,y' + c\,y = f\,(t)\,, \quad (\ell_2).$$

To solve (ℓ_1),

- assume a solution $y\,(t) = e^{r\,t}$ for some constant r.
- substitute $y\,(t) = e^{r\,t}$ into (ℓ_1) to get

$$a\,r^2\,e^{r\,t} + b\,r\,e^{r\,t} + c\,e^{r\,t} = 0 \iff a\,r^2 + b\,r + c = 0.$$

- possible solutions for r

$$r = r_{1,2} \stackrel{def}{=} \frac{-b \pm \sqrt{b^2 - 4\,a\,c}}{2\,a}.$$

- possible solutions for $y\,(t)$

$$y\,(t) = e^{r_1\,t},\ e^{r_2\,t}.$$

Homogeneous linear equations: **Example**

Find all solutions to

$$y'' + 5\,y' - 6\,y = 0, \quad (\ell_1)$$

SOLUTION: Roots to

$$r^2 + 5\,r - 6 = 0$$

are $r = 1, -6$, leading to solutions $y_1(t) = e^t,\ y_2(t) = e^{-6\,t}$.

$$y_1'' + 5\,y_1' - 6\,y_1 = 0, \quad y_2'' + 5\,y_2' - 6\,y_2 = 0, \quad (\ell_2)$$

- Let $y(t) = c_1\,y_1(t) + c_2\,y_2(t) \in \textbf{Span}\{y_1(t),\, y_2(t)\}$. Then

$$y''+5\,y'-6\,y = c_1\left(y_1'' + 5\,y_1' - 6\,y_1\right)+c_2\left(y_2'' + 5\,y_2' - 6\,y_2\right) \stackrel{by\ \ (\ell_2)}{=\!=\!=} 0.$$

- So ANY function in **Span** $\{y_1(t),\, y_2(t)\}$ is a solution to (ℓ_1)

Need additional conditions for unique solution

Solve Initial Value Problem (IVP): **Example**

$$y'' + 5\,y' - 6\,y = 0, \quad y\,(0) = 0, \quad y'\,(0) = -1, \quad (\ell_1)$$

SOLUTION: DE has solutions in the form $y\,(t) = c_1\,e^t + c_2\,e^{-6\,t}$.

$$\text{So} \quad y'\,(t) = c_1\,e^t - 6\,c_2\,e^{-6\,t}, \quad \text{leading to}$$

$$\begin{aligned} y\,(0) &= c_1 + c_2 = 0, \\ y'\,(0) &= c_1 - 6\,c_2 = -1. \end{aligned}$$

With solution $c_1 = -\frac{1}{7}$, $c_2 = \frac{1}{7}$. So solution to IVP

$$y\,(t) = -\frac{1}{7}\,e^t + \frac{1}{7}\,e^{-6\,t}.$$

Existence and Uniqueness: Homogeneous Case

Thm 1. For any real numbers a, b, c, t_0, Y_0, and Y_1 with $a \neq 0$, there exists a unique solution to IVP

$$a\,y'' + b\,y' + c\,y = 0, \quad y\,(t_0) = Y_0, \quad y'\,(t_0) = Y_1. \quad (\ell_1)$$

The solution is valid for all $t \in (-\infty, +\infty)$.

Existence and Uniqueness: Homogeneous Case

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The solution is valid for all $t \in (-\infty, +\infty)$.

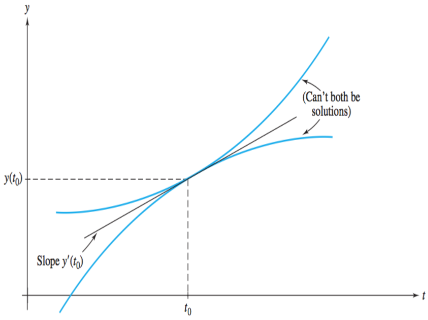

Figure 4.6 $y(t_0)$, $y'(t_0)$ determine a *unique* solution

Linear Independence of Two Functions

Definition 1. Functions $y_1(t)$ and $y_2(t)$ are

- **linearly independent** on the interval $I \iff$ neither of them is a constant multiple of the other on I,
- **linearly dependent** on I Otherwise.

EXAMPLE Let $y_1(t) = \sin(t)$ and $y_2(t) = |\sin(t)|$. Then $y_1(t)$ and $y_2(t)$ are

- **linearly independent** on the interval $(-\pi, +\pi)$,
- but **linearly dependent** on the interval $(0, +\pi)$.

Representation of Solutions to IVP

Thm 2. For any real numbers a, b, c, with $a \neq 0$, let $y_1(t)$ and $y_2(t)$ be two linearly independent solutions to

$$a\,y'' + b\,y' + c\,y = 0.$$

Then there exist unique constants c_1 and c_2 so that $y(t) = c_1\,y_1(t) + c_2\,y_2(t)$ satisfies the initial conditions

$$y(t_0) = Y_0, \quad y'(t_0) = Y_1.$$

The solution is valid for all $t \in (-\infty, +\infty)$.

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$$y(t_0) = Y_0, \quad y'(t_0) = Y_1.$$

The solution is valid for all $t \in (-\infty, +\infty)$.

Will prove **Thm 2.** with **Thm 1.**

Condition for Linear Dependence of Solutions

Lemma 1. For any real numbers a, b, c, with $a \neq 0$, let $y_1(t)$ and $y_2(t)$ be two solutions to

$$a\,y'' + b\,y' + c\,y = 0$$

that satisfy at any point τ

$$\mathbf{det}\begin{pmatrix} y_1(\tau) & y_2(\tau) \\ y_1'(\tau) & y_2'(\tau) \end{pmatrix} = y_1(\tau)\,y_2'(\tau) - y_1'(\tau)\,y_2(\tau) = 0$$

$$\Uparrow$$

(Wronskian of $y_1(t)$ and $y_2(t)$)

then $y_1(t)$ and $y_2(t)$ must be linearly dependent on $(-\infty, +\infty)$.

Lemma 1. $y_1(t)$ and $y_2(t)$ are solutions to

$$a\,y'' + b\,y' + c\,y = 0 \quad (\ell)$$

that satisfy $y_1(\tau)\;y_2'(\tau) - y_1'(\tau)\;y_2(\tau) = 0$. Need to show

$\boxed{y_1(t) \text{ and } y_2(t) \text{ must be linearly dependent on } (-\infty, +\infty)\,.}$

Lemma 1. $y_1(t)$ and $y_2(t)$ are solutions to

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$y_1(t)$ and $y_2(t)$ must be linearly dependent on $(-\infty, +\infty)$.

PROOF:

- If $y_1(\tau) \neq 0$, then $y_3(t) \stackrel{def}{=} \left(\frac{y_2(\tau)}{y_1(\tau)}\right) y_1(t)$ is solution to (ℓ),

$$y_3(\tau) = \left(\frac{y_2(\tau)}{y_1(\tau)}\right) y_1(\tau) = y_2(\tau), \quad y_3'(\tau) = \left(\frac{y_2(\tau)}{y_1(\tau)}\right) y_1'(\tau) = y_2'(\tau).$$

Lemma 1. $y_1(t)$ and $y_2(t)$ are solutions to

$$a\,y'' + b\,y' + c\,y = 0 \quad (\ell)$$

that satisfy $y_1(\tau)\;y_2'(\tau) - y_1'(\tau)\;y_2(\tau) = 0$. Need to show

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$$y_3(\tau) = \left(\frac{y_2(\tau)}{y_1(\tau)}\right) y_1(\tau) = y_2(\tau)\,, \quad y_3'(\tau) = \left(\frac{y_2(\tau)}{y_1(\tau)}\right) y_1'(\tau) = y_2'(\tau)\,.$$

- $y_3(t)$ and $y_2(t)$ satisfy the same initial conditions at τ.
  By **Thm. 1** they must be same on $(-\infty, +\infty)$.
  $\Longrightarrow \quad y_2(t)$ is a constant multiple of $y_1(t)$.

Lemma 1. $y_1(t)$ and $y_2(t)$ are solutions to

$$a\,y'' + b\,y' + c\,y = 0 \quad (\ell)$$

that satisfy $y_1(\tau)\, y_2'(\tau) - y_1'(\tau)\, y_2(\tau) = 0$. Need to show

$y_1(t)$ and $y_2(t)$ must be linearly dependent on $(-\infty, +\infty)$.

PROOF:

- If $y_1(\tau) \neq 0$, then $y_3(t) \overset{def}{=} \left(\frac{y_2(\tau)}{y_1(\tau)}\right) y_1(t)$ is solution to (ℓ),

$$y_3(\tau) = \left(\frac{y_2(\tau)}{y_1(\tau)}\right) y_1(\tau) = y_2(\tau), \quad y_3'(\tau) = \left(\frac{y_2(\tau)}{y_1(\tau)}\right) y_1'(\tau) = y_2'(\tau).$$

- $y_3(t)$ and $y_2(t)$ satisfy the same initial conditions at τ.
  By **Thm. 1** they must be same on $(-\infty, +\infty)$.
  $\Longrightarrow \quad y_2(t)$ is a constant multiple of $y_1(t)$.
- See book for case $y_1(\tau) = 0 \quad \square$.

Thm 2. Let $y_1(t)$ and $y_2(t)$ be linearly independent solutions to

$$a\,y'' + b\,y' + c\,y = 0. \quad (\ell_1)$$

Then there exist unique constants c_1 and c_2 so that $y(t) = c_1\,y_1(t) + c_2\,y_2(t)$ satisfies the initial conditions

$$y(t_0) = Y_0, \quad y'(t_0) = Y_1. \quad (\ell_2)$$

Thm 2. Let $y_1(t)$ and $y_2(t)$ be linearly independent solutions to

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Then there exist unique constants c_1 and c_2 so that $y(t) = c_1\,y_1(t) + c_2\,y_2(t)$ satisfies the initial conditions

$$y(t_0) = Y_0, \quad y'(t_0) = Y_1. \quad (\ell_2)$$

PROOF: $y(t)$ satisfies (ℓ_1) for any c_1, c_2. (ℓ_2) equivalent to

$$\begin{pmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} Y_0 \\ Y_1 \end{pmatrix},$$

which has a unique solution in c_1, c_2, since the coefficient matrix is invertible by **Lemma 1**:

$$\mathbf{det}\begin{pmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{pmatrix} \neq 0.$$

Distinct Real Roots

- Homogeneous linear 2^{nd} order constant-coefficient DE ($a \neq 0$)

$$a\,y'' + b\,y' + c\,y = 0, \quad (\ell_1),$$

- distinct real roots in equation $a\,r^2 + b\,r + c = 0$.

$$r_1,\ r_2 \stackrel{def}{=} \frac{-b \pm \sqrt{b^2 - 4\,a\,c}}{2\,a}.$$

- Linearly independent real solutions to (ℓ_1)

$$y_1(t) = e^{r_1\,t},\ \ y_2(t) = e^{r_2\,t}.$$

$$\mathbf{det}\begin{pmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{pmatrix} = (r_2 - r_1)\,e^{(r_2+r_1)\,t} \neq 0.$$

Distinct Real Roots: **Example**

- IVP

$$y'' + 5\,y' - 6\,y = 0, \quad y\,(0) = 0, \quad y'\,(0) = -1, \quad (\ell_1)$$

- distinct real roots $r_1 = 1, r_2 = -6$ to equation $r^2 + 5\,r - 6 = 0$.
- (ℓ_1) has solutions in the form $y\,(t) = c_1\,e^t + c_2\,e^{-6\,t}$.

  So $\quad y'\,(t) = c_1\,e^t - 6\,c_2\,e^{-6\,t}, \quad$ Initial conditions lead to

$$\begin{aligned} y\,(0) &= c_1 + c_2 = 0, \\ y'\,(0) &= c_1 - 6\,c_2 = -1. \end{aligned}$$

  With solution $c_1 = -\frac{1}{7}$, $c_2 = \frac{1}{7}$.
- So solution to IVP

$$y\,(t) = -\frac{1}{7}\,e^t + \frac{1}{7}\,e^{-6\,t}.$$

Double Real Root

- Homogeneous linear 2^{nd} order constant-coefficient DE ($a \neq 0$)

$$a\, y'' + b\, y' + c\, y = 0, \quad (\ell_1),$$

- double real root in equation $a\, r^2 + b\, r + c = 0.$

$$r_1 \stackrel{def}{=} -\frac{b}{2\, a}, \quad \text{with} \quad b^2 - 4\, a\, c = 0.$$

- One solution to (ℓ_1)
- $y_1(t) = e^{r_1 t}$ is solution to (ℓ_1); another is $y_2(t) = t\, e^{r_1 t}$:

$$\begin{aligned} y_2'(t) &= e^{r_1 t} + r_1\, y_2(t), \quad y_2''(t) = 2\, r_1\, e^{r_1 t} + r_1^2\, y_2(t) \\ a\, y_2'' + b\, y_2' + c\, y_2 &= (2\, r_1\, a + b)\, e^{r_1 t} + \left(a\, r_1^2 + b\, r_1 + c\right) y_2(t) = 0. \end{aligned}$$

- $y_1(t)$ and $y_2(t)$ are linearly independent

$$\mathbf{det} \begin{pmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{pmatrix} = e^{(2\, r_1)\, t} \neq 0.$$

Double Real Roots: **Example**

- IVP

$$y'' + 4\,y' + 4\,y = 0, \quad y\,(0) = 0, \quad y'\,(0) = -1, \quad (\ell_1)$$

- double real root $r_1 = r_2 = -2$ to equation $r^2 + 4\,r + 4 = 0$.
- (ℓ_1) has solutions in the form $y\,(t) = c_1\,e^{-2\,t} + c_2\,t\,e^{-2\,t}$.

  So $\quad y'\,(t) = -2\,c_1\,e^{-2\,t} + \,c_2\,\left(e^{-2\,t} - 2\,t\,e^{-2\,t}\right),$

- Initial conditions lead to

$$\begin{aligned} y\,(0) &= c_1 &&= 0, \\ y'\,(0) &= -2\,c_1 + c_2 &&= -1. \end{aligned}$$

  With solution $c_1 = 0$, $c_2 = -1$.
- So solution to IVP

$$y\,(t) = -t\,e^{-2\,t}.$$

§4.3 Complex Conjugate Roots (I)

- Homogeneous linear 2^{nd} order constant-coefficient DE ($a \neq 0$)

$$a\,y'' + b\,y' + c\,y = 0, \quad \text{with} \quad b^2 - 4\,a\,c < 0 \quad (\ell_1),$$

- complex conjugate roots in $a\,r^2 + b\,r + c = 0$ with $i^2 = -1$:

$$r_1,\ r_2 = \frac{-b \pm \sqrt{b^2 - 4\,a\,c}}{2\,a} \stackrel{def}{=} \alpha \pm i\,\beta, \quad \text{with} \ \ \alpha = -\frac{b}{2\,a}, \ \ \beta = \frac{\sqrt{4\,a\,c - b^2}}{2\,a}.$$

- Linearly independent $\boxed{\text{complex}}$ solutions to (ℓ_1)

$$y_1(t) = e^{(\alpha + i\,\beta)\,t}, \ \ y_2(t) = e^{(\alpha - i\,\beta)\,t}.$$

$$\mathbf{det}\begin{pmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{pmatrix} = -2\,i\,\beta\,e^{2\,\alpha\,t} \neq 0.$$

Complex Conjugate Roots (II)

- Homogeneous linear 2^{nd} order constant-coefficient DE ($a \neq 0$)

$$a\,y'' + b\,y' + c\,y = 0, \quad \text{with} \quad b^2 - 4\,a\,c < 0 \quad (\ell_1),$$

- Linearly independent $\boxed{\text{complex}}$ solutions to (ℓ_1)

$$y_1(t) = e^{(\alpha + i\,\beta)\,t}, \;\; y_2(t) = e^{(\alpha - i\,\beta)\,t}, \quad \alpha = -\frac{b}{2\,a}, \;\; \beta = \frac{\sqrt{4\,a\,c - b^2}}{2\,a}.$$

- **Euler's formula**: $e^{(\alpha \pm i\,\beta)\,t} = e^{\alpha\,t}\left(\cos(\beta\,t) \pm i \sin(\beta\,t)\right).$
- Linearly independent $\boxed{\text{real}}$ solutions to (ℓ_1)

$$\begin{aligned} \widehat{y}_1(t) &= \frac{1}{2}\left(y_1(t) + y_2(t)\right) = e^{\alpha\,t}\cos(\beta\,t), \\ \widehat{y}_2(t) &= \frac{1}{2i}\left(y_1(t) - y_2(t)\right) = e^{\alpha\,t}\sin(\beta\,t). \end{aligned}$$

Complex Conjugate Roots: **Example**

- IVP

$$y'' - 4\,y' + 5\,y = 0, \quad y\,(0) = 0, \quad y'\,(0) = -1, \quad (\ell_1)$$

- complex conjugate roots $r_1, r_2 = 2 \pm\ i$ to $r^2 - 4\,r + 5 = 0$.
- (ℓ_1) has solutions $y\,(t) = e^{2\,t}\,(c_1\,\mathbf{cos}\,(t) + c_2\,\mathbf{sin}\,(t))$.

  So $\quad y'\,(t) = e^{2\,t}\,((2\,c_1 + c_2)\,\mathbf{cos}\,(t) + (2\,c_2 - c_1)\,\mathbf{sin}\,(t))$

- Initial conditions lead to

$$\begin{aligned} y\,(0) &= c_1 &&= 0, \\ y'\,(0) &= 2\,c_1 + c_2 &&= -1. \end{aligned}$$

  With solution $c_1 = 0,\ c_2 = -1$.
- So solution to IVP

$$y\,(t) = -e^{2\,t}\,\mathbf{sin}\,(t)\,.$$

Damped mass–spring oscillator: **Example** (I)

$$\underbrace{m}_{\textbf{inertia}} \times y'' \quad + \quad \underbrace{b}_{\textbf{damping}} \times y' \quad + \quad \underbrace{k}_{\textbf{stiffness}} \times y \quad = \quad \mathbf{F_{ext}}(t),$$

- Determine the motion when

$m = 36\text{kg}, b = 12\text{kg/sec}, k = 37\text{kg/sec}^2, y(0) = 0.7\text{m}, y'(0) = 0.1\text{m/sec}.$

SOLUTION: IVP is

$$36\,y'' + 12\,y' + 37\,y = 0, \quad y(0) = 0.7,\ y'(0) = 0.1.$$

Roots to $36\,r^2 + 12\,r + 37 = 0$ are $r = -\frac{1}{6} \pm i$, so

$$\begin{aligned} y(t) &= e^{-\frac{t}{6}}\left(c_1 \cos(t) + c_2 \sin(t)\right), & y(0) &= c_1 \\ y'(t) &= e^{-\frac{t}{6}}\left(\left(c_2 - \frac{c_1}{6}\right)\cos(t) - \left(c_1 + \frac{c_2}{6}\right)\sin(t)\right), & y'(0) &= c_2 - \frac{c_1}{6}. \end{aligned}$$

Initial conditions lead to $c_1 = 0.7,\ c_2 = 1.3/6.$

$$y(t) = e^{-\frac{t}{6}}\left(0.7\cos(t) + 1.3/6\sin(t)\right).$$

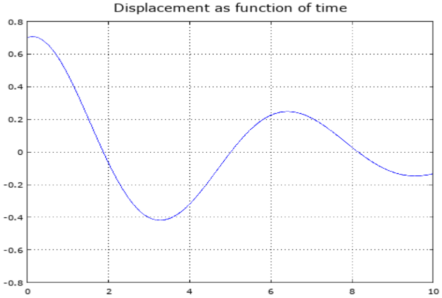
Damped mass–spring oscillator: **Example** (II)

- After how many seconds will the mass first cross the equilibrium point?

SOLUTION: Solution to IVP is

$$\begin{aligned} y(t) &= e^{-\frac{t}{6}} \left(0.7\,\mathbf{cos}(t) + 1.3/6\,\mathbf{sin}(t)\right) \\ &= \sqrt{0.7^2 + (1.3/6)^2}\, e^{-\frac{t}{6}}\,\mathbf{sin}(t + t_0), \quad \mathbf{sin}(t_0) \stackrel{def}{=} \frac{0.7}{\sqrt{0.7^2 + (1.3/6)^2}}. \end{aligned}$$

Setting $y(t) = 0$ gives $t = \pi - t_0 \approx 1.87$(sec).



§4.4 Nonhomogeneous equations

- Nonhomogeneous linear 2^{nd} order constant-coefficient differential equation

$$a\,y'' + b\,y' + c\,y = f(t)\,, \quad a \neq 0, \quad (\ell)$$

- Solve (ℓ) for specific types of $f(t)$.
- Focus on one PARTICULAR solution for each $f(t)$ for now.

Nonhomogeneous equations, **Example** (I)

Find one PARTICULAR solution for

$$y'' + 3\,y' + 2\,y = 3\,t$$

SOLUTION:

- Assume a solution of form $y\,(t) = A\,t + B$,

$$y' = A, \;\; y'' = 0,$$

$$\text{and} \quad y'' + 3\,y' + 2\,y = 3\,A + 2\,(A\,t + B) = 3\,t$$

- Which implies $2\,A = 3,\; 3\,A + 2\,B = 0$
- Therefore $A = \frac{3}{2},\; B = -\frac{9}{4}$ and

$$y\,(t) = \frac{3}{2}\,t - \frac{9}{4}.$$

Nonhomogeneous equations, **Example** (II)

Find one PARTICULAR solution for

$$y'' + 3\,y' + 2\,y = 10\,e^{3\,t}$$

SOLUTION:

- Assume a solution of form $y\,(t) = A\;e^{3\,t}$,

$$y' = 3\,A\,e^{3\,t},\;\; y'' = 3^2\,A\,e^{3\,t}, \quad \text{and}$$

$$y'' + 3\,y' + 2\,y = 3^2\,A\,e^{3\,t} + 3 \cdot 3\,A\,e^{3\,t} + 2 \cdot\,A\,e^{3\,t} = 20\,A\,e^{3\,t} = 10\,e^{3\,t}$$

- Therefore $A = \frac{1}{2}$ and

$$y\,(t) = \frac{1}{2}\,e^{3\,t}.$$

Nonhomogeneous equations, **Example** (III)

Find one PARTICULAR solution for

$$y'' + 3\,y' + 2\,y = \sin(t)$$

SOLUTION:

- Assume a solution of form $y(t) = A\sin(t) + B\cos(t)$, then

$$y' = A\cos(t) - B\sin(t)\,, \quad y'' = -A\sin(t) - B\cos(t)\,, \text{ and}$$

$$\begin{aligned} y'' + 3\,y' + 2\,y &= -\left(A\sin(t) + B\cos(t)\right) + 3\,\left(A\cos(t) - B\sin(t)\right) \\ &\quad +2\,\left(A\sin(t) + B\cos(t)\right) \\ &= (A - 3\,B)\,\sin(t) + (B + 3\,A)\,\cos(t) = \sin(t)\,, \end{aligned}$$

- which implies $A - 3\,B = 1,\; B + 3\,A = 0.$
- Therefore $A = \frac{1}{10},\; B = -\frac{3}{10}$ and

$$y(t) = \frac{1}{10}\left(\sin(t) - 3\cos(t)\right).$$

Nonhomogeneous equations, **Example** (IV)

Find one PARTICULAR solution for

$$y'' + 4\,y = 5\,t^2\,e^t$$

SOLUTION:

- Assume a solution of form $y\,(t) = \left(A\,t^2 + B\,t + C\right)\,e^t$,

$$\begin{aligned} y' &= (2\,A\,t + B)\,e^t + \left(A\,t^2 + B\,t + C\right)\,e^t, \\ y'' &= 2\,A\,e^t + 2\,(2\,A\,t + B)\,e^t + \left(A\,t^2 + B\,t + C\right)\,e^t \end{aligned}$$

and

$$\begin{aligned} y'' + 4\,y &= 2\,A\,e^t + 2\,(2\,A\,t + B)\,e^t + 5\left(A\,t^2 + B\,t + C\right)\,e^t \\ &= \left(5\,A\,t^2 + (4\,A + 5\,B)\,t + (2\,A + 2\,B + 5\,C)\right)\,e^t \end{aligned}$$

- which implies $5\,A = 5$, $4\,A + 5\,B = 0$, $2\,A + 2\,B + 5\,C = 0$.
- Therefore $A = 1$, $B = -\frac{4}{5}$, $C = -\frac{2}{25}$, and

$$y\,(t) = \left(t^2 - \frac{4}{5}\,t - \frac{2}{25}\right)\,e^t.$$

Nonhomogeneous equations, **Example** (V)

Find one PARTICULAR solution for

$$y'' + y' = 5\,t$$

SOLUTION:

- Assume a solution of form $y(t) = A\,t^2 + B\,t + C$,

$$y' = 2\,A\,t + B, \quad y'' = 2\,A.$$

and

$$y'' + y' = 2\,A + 2\,A\,t + B = 5\,t$$

- which implies $2\,A = 5$, $2\,A + B = 0$.
- Therefore $A = \frac{5}{2}$, $B = -5$, and

$$y(t) = \frac{5}{2}\,t^2 - 5\,t.$$

Nonhomogeneous equations: General case (I)

Find one PARTICULAR solution for integer $m \geq 0$,

$$a\,y'' + b\,y' + c\,y = t^m\,e^{r\,t}, \quad (a \neq 0) \quad (\ell_1)$$

Nonhomogeneous equations: General case (I)

Find one PARTICULAR solution for integer $m \geq 0$,

$$a\,y'' + b\,y' + c\,y = t^m\,e^{r\,t}, \quad (a \neq 0) \quad (\ell_1)$$

SOLUTION:

- Assume a solution of form $y\,(t) = e^{r\,t}\widehat{y}\,(t)$,

$$\begin{aligned} y' &= r\,e^{r\,t}\widehat{y}\,(t) + e^{r\,t}\widehat{y}'\,(t)\,, \\ y'' &= r^2\,e^{r\,t}\widehat{y}\,(t) + 2\,r\,e^{r\,t}\widehat{y}'\,(t) + e^{r\,t}\widehat{y}''\,(t)\,, \quad \text{and} \end{aligned}$$

$$\begin{aligned} a\,y'' + b\,y' + c\,y &= \left(a\,r^2 + b\,r + c\right)\,e^{r\,t}\widehat{y}\,(t) + (2\,a\,r + b)\,e^{r\,t}\widehat{y}'\,(t) \\ &\quad + a\,e^{r\,t}\,\widehat{y}''\,(t) \end{aligned}$$

Nonhomogeneous equations: General case (I)

Find one PARTICULAR solution for integer $m \geq 0$,

$$a\,y'' + b\,y' + c\,y = t^m\,e^{r\,t}, \quad (a \neq 0) \quad (\ell_1)$$

SOLUTION:

- Assume a solution of form $y(t) = e^{r\,t}\widehat{y}(t)$,

$$\begin{aligned} y' &= r\,e^{r\,t}\widehat{y}(t) + e^{r\,t}\widehat{y}'(t), \\ y'' &= r^2\,e^{r\,t}\widehat{y}(t) + 2\,r\,e^{r\,t}\widehat{y}'(t) + e^{r\,t}\widehat{y}''(t), \quad \text{and} \end{aligned}$$

$$\begin{aligned} a\,y'' + b\,y' + c\,y &= \left(a\,r^2 + b\,r + c\right)\,e^{r\,t}\widehat{y}(t) + (2\,a\,r + b)\,e^{r\,t}\widehat{y}'(t) \\ &\quad + a\,e^{r\,t}\,\widehat{y}''(t) \end{aligned}$$

- equation (ℓ_1) becomes

$$\left(a\,r^2 + b\,r + c\right)\widehat{y}(t) + (2\,a\,r + b)\,\widehat{y}'(t) + a\,\widehat{y}''(t) = t^m. \quad (\ell_2)$$

Nonhomogeneous equations: General case (I)

Find one PARTICULAR solution for integer $m \geq 0$,

$$a\,y'' + b\,y' + c\,y = t^m\,e^{r\,t}, \quad (a \neq 0) \quad (\ell_1)$$

SOLUTION:

- Assume a solution of form $y\,(t) = e^{r\,t}\widehat{y}\,(t)$,

$$\begin{aligned} y' &= r\,e^{r\,t}\widehat{y}\,(t) + e^{r\,t}\widehat{y}'\,(t)\,, \\ y'' &= r^2\,e^{r\,t}\widehat{y}\,(t) + 2\,r\,e^{r\,t}\widehat{y}'\,(t) + e^{r\,t}\widehat{y}''\,(t)\,, \quad \text{and} \end{aligned}$$

$$\begin{aligned} a\,y'' + b\,y' + c\,y &= \left(a\,r^2 + b\,r + c\right)\,e^{r\,t}\widehat{y}\,(t) + (2\,a\,r + b)\,e^{r\,t}\widehat{y}'\,(t) \\ &\quad + a\,e^{r\,t}\,\widehat{y}''\,(t) \end{aligned}$$

- equation (ℓ_1) becomes

$$\left(a\,r^2 + b\,r + c\right)\widehat{y}\,(t) + (2\,a\,r + b)\,\widehat{y}'\,(t) + a\,\widehat{y}''\,(t) = t^m. \quad (\ell_2)$$

- Choose $\widehat{y}\,(t) = t^s\,\left(A_0 + A_1\,t + \cdots + A_m\,t^m\right)$ to satisfy (ℓ_2),

$$s = \begin{cases} 0, & \text{if} \quad a\,r^2 + b\,r + c \neq 0, \\ 1, & \text{if} \quad a\,r^2 + b\,r + c = 0,\ 2\,a\,r + b \neq 0, \\ 2, & \text{if} \quad a\,r^2 + b\,r + c = 0,\ 2\,a\,r + b = 0. \end{cases}$$

Method of undetermined coefficients

Find one PARTICULAR solution for

$$y'' - 2\,y' + y = t^2\,e^t$$

SOLUTION:

- equation $r^2 - 2\,r + 1 = 0$ has double root $r = 1$.
- Must try solution $y\,(t) = t^2\,\left(A_0 + A_1\,t + A_2\,t^2\right)\,e^t$ such that

$$\frac{d^2}{d\,t^2}\left(t^2\,\left(A_0 + A_1\,t + A_2\,t^2\right)\right) = t^2,$$

- Therefore $A_0 = A_1 = 0, A_2 = \frac{1}{12}$, and

$$y\,(t) = \frac{1}{12}\,t^4\,e^t.$$

§4.5 Superposition principle

Thm. 3: Let y_1 be a solution to the differential equation

$$a\,y'' + b\;y' + c\,y = f_1\,(t)\,,\quad \text{and}$$

$$y_2 \;\text{ be solution to }\quad a\,y'' + b\;y' + c\,y = f_2\,(t)\,.$$

Then for any constants k_1 and k_2, the function $k_1\,y_1 + k_2\,y_2$ is a solution to the differential equation

$$a\,y'' + b\;y' + c\,y = k_1\,f_1\,(t) + k_2\,f_2\,(t)\,,$$

§4.5 Superposition principle

Thm. 3: Let y_1 be a solution to the differential equation

$$a\,y'' + b\,y' + c\,y = f_1(t)\,, \quad \text{and}$$

$$y_2 \ \text{ be solution to } \quad a\,y'' + b\,y' + c\,y = f_2(t)\,.$$

Then for any constants k_1 and k_2, the function $k_1\,y_1 + k_2\,y_2$ is a solution to the differential equation

$$a\,y'' + b\,y' + c\,y = k_1\,f_1(t) + k_2\,f_2(t)\,,$$

PROOF: This is a simple substitution:

$$\begin{aligned} & a\,(k_1\,y_1 + k_2\,y_2)'' + b\,(k_1\,y_1 + k_2\,y_2)' + c\,(k_1\,y_1 + k_2\,y_2) \\ = \; & k_1\left(a\,y_1'' + b\,y_1' + c\,y_1\right) + k_2\left(a\,y_2'' + b\,y_2' + c\,y_2\right) \\ = \; & k_1\,f_1(t) + k_2\,f_2(t)\,. \end{aligned}$$

Superposition, **Example** (I)

Find one PARTICULAR solution for

$$y'' - 2\,y' + y = 5\,t^2\,e^t - 2\,e^{2\,t} \quad (\ell)$$

SOLUTION:

- $y_1(t) = \frac{t^4\,e^t}{12}$ is solution to

$$y'' - 2\,y' + y = t^2\,e^t$$

- $y_2(t) = e^{2\,t}$ is solution to

$$y'' - 2\,y' + y = e^{2\,t}$$

- Therefore

$$y(t) = 5\,y_1(t) - 2\,y_2(t) = \frac{5\,t^4\,e^t}{12} - 2\,e^{2\,t}$$

is solution to (ℓ).

Nonhomogeneous equations: General case (II)

Find one PARTICULAR solution for integer $m \geq 0$, $a \neq 0$,

$$a\,y'' + b\,y' + c\,y = t^m\,e^{\alpha\,t}\,\mathbf{cos}\,(\beta\,t) \qquad (\ell_1)$$

Nonhomogeneous equations: General case (II)

Find one PARTICULAR solution for integer $m \geq 0$, $a \neq 0$,

$$a\,y'' + b\,y' + c\,y = t^m\,e^{\alpha\,t}\,\mathbf{cos}\,(\beta\,t) \quad (\ell_1)$$

SOLUTION: Let $r = \alpha + i\,\beta$

- Assume a solution of form $y\,(t) = e^{r\,t}\widehat{y}\,(t)$,

$$\begin{aligned} y' &= r\,e^{r\,t}\widehat{y}\,(t) + e^{r\,t}\widehat{y}'\,(t)\,, \\ y'' &= r^2\,e^{r\,t}\widehat{y}\,(t) + 2\,r\,e^{r\,t}\widehat{y}'\,(t) + e^{r\,t}\widehat{y}''\,(t)\,, \quad \text{and} \end{aligned}$$

$$\begin{aligned} a\,y'' + b\,y' + c\,y &= \left(a\,r^2 + b\,r + c\right)\,e^{r\,t}\widehat{y}\,(t) + (2\,a\,r + b)\,e^{r\,t}\widehat{y}'\,(t) \\ &\quad + a\,e^{r\,t}\,\widehat{y}''\,(t) \end{aligned}$$

Nonhomogeneous equations: General case (II)

Find one PARTICULAR solution for integer $m \geq 0$, $a \neq 0$,

$$a\,y'' + b\,y' + c\,y = t^m\, e^{\alpha\, t}\, \mathbf{cos}\,(\beta\, t) \qquad (\ell_1)$$

SOLUTION: Let $r = \alpha + i\,\beta$

- Assume a solution of form $y\,(t) = e^{r\,t}\widehat{y}\,(t)$,

$$\begin{aligned} y' &= r\,e^{r\,t}\widehat{y}\,(t) + e^{r\,t}\widehat{y}'\,(t)\,, \\ y'' &= r^2\,e^{r\,t}\widehat{y}\,(t) + 2\,r\,e^{r\,t}\widehat{y}'\,(t) + e^{r\,t}\widehat{y}''\,(t)\,, \quad \text{and} \end{aligned}$$

$$\begin{aligned} a\,y'' + b\,y' + c\,y &= \left(a\,r^2 + b\,r + c\right)\,e^{r\,t}\widehat{y}\,(t) + (2\,a\,r + b)\,e^{r\,t}\widehat{y}'\,(t) \\ &\quad + a\,e^{r\,t}\,\widehat{y}''\,(t) \end{aligned}$$

- equation (ℓ_1) becomes

$$\left(a\,r^2 + b\,r + c\right)\widehat{y}\,(t) + (2\,a\,r + b)\,\widehat{y}'\,(t) + a\,\widehat{y}''\,(t) = t^m. \qquad (\ell_2)$$

Nonhomogeneous equations: General case (II)

Find one PARTICULAR solution for integer $m \geq 0$, $a \neq 0$,

$$a\,y'' + b\,y' + c\,y = t^m\, e^{\alpha\, t} \cos(\beta\, t) \quad (\ell_1)$$

SOLUTION: Let $r = \alpha + i\,\beta$

- Assume a solution of form $y(t) = e^{r\,t}\widehat{y}(t)$,

$$\begin{aligned} y' &= r\,e^{r\,t}\widehat{y}(t) + e^{r\,t}\widehat{y}'(t), \\ y'' &= r^2\,e^{r\,t}\widehat{y}(t) + 2\,r\,e^{r\,t}\widehat{y}'(t) + e^{r\,t}\widehat{y}''(t), \quad \text{and} \end{aligned}$$

$$\begin{aligned} a\,y'' + b\,y' + c\,y &= \left(a\,r^2 + b\,r + c\right)\, e^{r\,t}\widehat{y}(t) + (2\,a\,r + b)\, e^{r\,t}\widehat{y}'(t) \\ &\quad + a\, e^{r\,t}\,\widehat{y}''(t) \end{aligned}$$

- equation (ℓ_1) becomes

$$\left(a\,r^2 + b\,r + c\right)\widehat{y}(t) + (2\,a\,r + b)\,\widehat{y}'(t) + a\,\widehat{y}''(t) = t^m. \quad (\ell_2)$$

- Choose $\widehat{y}(t) = t^s\,\left(A_0 + A_1\,t + \cdots + A_m\,t^m\right)$ to satisfy (ℓ_2),

$$s = \begin{cases} 0, & \text{if} \quad a\,r^2 + b\,r + c \neq 0, \\ 1, & \text{if} \quad a\,r^2 + b\,r + c = 0. \end{cases}$$

Nonhomogeneous equations: General case (III)

Find one PARTICULAR solution for integer $m \geq 0$, $a \neq 0$,

$$a\, y'' + b\, y' + c\, y = P_m(t)\; e^{\alpha\, t} \cos(\beta\, t) + Q_m(t)\; e^{\alpha\, t} \sin(\beta\, t)\,, \qquad (\ell)$$

where $P_m(t)$ and $Q_m(t)$ are polynomials of degree $\leq m$.

Nonhomogeneous equations: General case (III)

Find one PARTICULAR solution for integer $m \geq 0$, $a \neq 0$,

$$a\,y'' + b\,y' + c\,y = P_m(t)\; e^{\alpha t} \cos(\beta t) + Q_m(t)\; e^{\alpha t} \sin(\beta t)\,, \qquad (\ell)$$

where $P_m(t)$ and $Q_m(t)$ are polynomials of degree $\leq m$.
SOLUTION: Let $r = \alpha + i\,\beta$

- Assume a solution of form

$$y(t) = t^s \left(\widehat{P}_m(t)\; e^{\alpha t} \cos(\beta t) + \widehat{Q}_m(t)\; e^{\alpha t} \sin(\beta t) \right), \quad \text{where}$$

$$\begin{aligned} \widehat{P}_m(t) &= A_0 + A_1\,t + \cdots + A_m\,t^m, \\ \widehat{Q}_m(t) &= B_0 + B_1\,t + \cdots + B_m\,t^m, \\ s &= \begin{cases} 0, & \text{if} \quad a\,r^2 + b\,r + c \neq 0, \\ 1, & \text{if} \quad a\,r^2 + b\,r + c = 0. \end{cases} \end{aligned}$$

Choose $A_0, \cdots, A_m, B_0, \cdots, B_m$ to satisfy (ℓ)

Superposition, **Example** (II)

Find one PARTICULAR solution for

$$y'' - 2\,y' + 2\,y = 5\,t\,e^{t}\,\mathbf{sin}\,(t) - 2\,e^{2\,t} \qquad (\ell)$$

SOLUTION:

- Let $y_1\,(t) = t\,\left(A_0 + A_1\,t\right) e^{t}\,\mathbf{cos}\,(t) + t\,\left(B_0 + B_1\,t\right) e^{t}\,\mathbf{sin}\,(t)$

  $$\text{be solution to} \quad y'' - 2\,y' + y = t\,e^{t}\,\mathbf{sin}\,(t)\,,$$

  leading to $y_1\,(t) = \frac{t}{4}\,e^{t}\,\left(\mathbf{sin}\,(t) - t\,\mathbf{cos}\,(t)\right)$
- $y_2\,(t) = \frac{1}{2}\,e^{2\,t}$ is solution to

  $$y'' - 2\,y' + 2\,y = e^{2\,t}$$

- Therefore

  $$y\,(t) = 5\,y_1\,(t) - 2\,y_2\,(t) = \frac{5}{4}\,t\,e^{t}\,\left(\mathbf{sin}\,(t) - t\,\mathbf{cos}\,(t)\right) - e^{2\,t}$$

  is solution to (ℓ).

Existence and Uniqueness: Nonhomogeneous Case (I)

Thm 4. For any real numbers a, b, c, t_0, Y_0, and Y_1 with $a \neq 0$, suppose $y_p(t)$ is a particular solution to

$$a\,y'' + b\,y' + c\,y = f(t) \quad (\ell_1)$$

in an interval I containing t_0 and that $y_1(t)$ and $y_2(t)$ are linearly independent solutions to

$$a\,y'' + b\,y' + c\,y = 0.$$

Then there exists a unique solution in I to (ℓ_1) in the form

$$y(t) = y_p(t) + c_1\,y_1(t) + c_2\,y_2(t)$$

that satisfies initial conditions

$$y(t_0) = Y_0, \quad y'(t_0) = Y_1. \quad (\ell_2)$$

Existence and Uniqueness: Nonhomogeneous Case (I)

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Then there exists a unique solution in I to (ℓ_1) in the form

$$y(t) = y_p(t) + c_1\,y_1(t) + c_2\,y_2(t)$$

that satisfies initial conditions

$$y(t_0) = Y_0, \quad y'(t_0) = Y_1. \quad (\ell_2)$$

PROOF: $\widehat{y}(t) \stackrel{def}{=} y(t) - y_p(t)$.

- $y(t)$ is solution in I to (ℓ_1) with initial conditions (ℓ_2) $\Longleftrightarrow$ $\widehat{y}(t)$ is solution to (ℓ_3)

$$a\,y''+b\,y'+c\,y = 0, \quad y(t_0) = Y_0 - y_p(t_0), \quad y'(t_0) = Y_1 - y_p'(t_0). \quad (\ell_3)$$

Existence and Uniqueness: Nonhomogeneous Case (II)

PROOF: $\widehat{y}(t) \stackrel{def}{=} y(t) - y_p(t)$.

- $y(t)$ is solution in I to (ℓ_1) with initial conditions (ℓ_2) $\Longleftrightarrow$ $\widehat{y}(t)$ is solution to (ℓ_3)

$$a\,y'' + b\,y' + c\,y = 0, \quad y(t_0) = Y_0 - y_p(t_0), \quad y'(t_0) = Y_1 - y_p'(t_0). \quad (\ell_3)$$

- $\widehat{y}(t) = c_1\,y_1(t) + c_2\,y_2(t)$ is solution to (ℓ_3) $\Longleftrightarrow$

$$\begin{pmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} Y_0 - y_p(t_0) \\ Y_1 - y_p'(t_0) \end{pmatrix}. \quad (\ell_4)$$

- By **Lemma 1**, linear independence of $y_1(t)$ and $y_2(t)$ implies

$$\mathbf{det}\begin{pmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{pmatrix} \neq 0.$$

Therefore, there is a unique solution in (ℓ_4), thus a unique solution in (ℓ_3).

Damped mass–spring oscillator with external force (I)

$$m \times y'' \quad + \quad b \times y' \quad + \quad k \times y \quad = \quad \mathbf{F_{ext}}(t)\,.$$

- Find motion for $\mathbf{F_{ext}}(t) = (5\,\mathbf{cos}(t) + 5\,\mathbf{sin}(t))$ kg*m/sec^2,

$m = 1$kg, $b = 2$kg/sec, $k = 2$kg/sec^2, $y(0) = 1$m, $y'(0) = 2$m/sec.

SOLUTION: IVP is

$$y'' + 2\,y' + 2\,y = 5\,\mathbf{cos}(t) + 5\,\mathbf{sin}(t)\,, \quad y(0) = 1,\; y'(0) = 2.$$

Roots to $r^2 + 2\,r + 2 = 0$ are $r = -1 \pm i$, so

- A particular solution takes form $y_p(t) = A\,\mathbf{cos}(t) + B\,\mathbf{sin}(t)\,.$

  $$\text{Setting} \quad y_p'' + 2\,y_p' + 2\,y_p = 5\,\mathbf{cos}(t) + 5\,\mathbf{sin}(t)$$

  leads to $y_p(t) = -\mathbf{cos}(t) + 3\,\mathbf{sin}(t)$.

- $\widehat{y}(t) \stackrel{def}{=} y(t) - y_p(t)$ satisfies

$$\widehat{y}'' + 2\,\widehat{y}' + 2\,\widehat{y} = 0,$$

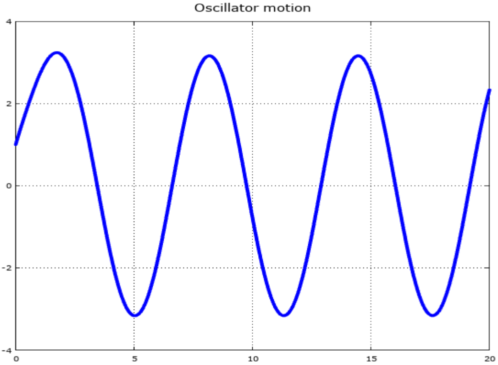
with initial conditions $\widehat{y}(0) = 1 - y_p(0) = 2, \quad \widehat{y}'(0) = 2 - y_p'(0) = -1.$

- IVP solution $\widehat{y}(t) = 2\,e^{-t}\,\mathbf{cos}(t) + e^{-t}\,\mathbf{sin}(t)$

Damped mass–spring oscillator with external force (II)

Oscillator motion

$$\begin{aligned} y(t) &= \widehat{y}(t) + y_p(t) \\ &= 2\,e^{-t}\,\mathbf{cos}(t) + e^{-t}\,\mathbf{sin}(t) - \mathbf{cos}(t) + 3\,\mathbf{sin}(t)\,. \end{aligned}$$



§4.6 Variation of parameters (I)

Let $y_1(t)$ and $y_2(t)$ be two linearly independent solutions for

$$a\,y'' + b\,y' + c\,y = 0. \quad (\ell_1)$$

We look for a particular solution to

$$a\,y'' + b\,y' + c\,y = f(t) \quad (\ell_2)$$

§4.6 Variation of parameters (I)

Let $y_1(t)$ and $y_2(t)$ be two linearly independent solutions for

$$a\,y'' + b\,y' + c\,y = 0. \quad (\ell_1)$$

We look for a particular solution to

$$a\,y'' + b\,y' + c\,y = f(t) \quad (\ell_2)$$

- For any constants $c_1,\ c_2,\ y_h(t) = c_1,\ y_1(t) + c_2\,y_2(t)$ is a solution to (ℓ_1).
- Find particular solution to (ℓ_2) in the form $y_p(t) = v_1(t)\ y_1(t) + v_2(t)\ y_2(t)$.

Will derive two equations that determine $v_1(t)$ and $v_2(t)$

$y_p(t) = v_1(t)\, y_1(t) + v_2(t)\, y_2(t)$ is particular solution to

$$a\, y_p'' + b\, y_p' + c\, y_p = f(t) \qquad (\ell_1)$$

$y_p(t) = v_1(t)\ y_1(t) + v_2(t)\ y_2(t)$ is particular solution to

$$a\,y_p'' + b\,y_p' + c\,y_p = f(t) \qquad (\ell_1)$$

$$y_p'(t) = v_1'(t)\ y_1(t) + v_2'(t)\ y_2(t) + v_1(t)\ y_1'(t) + v_2(t)\ y_2'(t),$$

Choose $\quad v_1'(t)\ y_1(t) + v_2'(t)\ y_2(t) = 0. \qquad (\ell_2)$

$$y_p''(t) = v_1'(t)\ y_1'(t) + v_2'(t)\ y_2'(t) + v_1(t)\ y_1''(t) + v_2(t)\ y_2''(t).$$

Equation (ℓ_1) is

$$\begin{aligned} f(t) &= a\left(v_1'(t)\ y_1'(t) + v_2'(t)\ y_2'(t) + v_1(t)\ y_1''(t) + v_2(t)\ y_2''(t)\right) + \\ &\quad b\left(v_1(t)\ y_1'(t) + v_2(t)\ y_2'(t)\right) + c\left(v_1(t)\ y_1(t) + v_2(t)\ y_2(t)\right) \end{aligned}$$

Re-arranging terms,

$$\begin{aligned} f(t) &= a\left(v_1'(t)\ y_1'(t) + v_2'(t)\ y_2'(t)\right) \\ &\quad + v_1(t)\left(a\,y_1'' + b\,y_1' + c\,y_1\right) \\ &\quad + v_2(t)\left(a\,y_2'' + b\,y_2' + c\,y_2\right) \\ &= a\left(v_1'(t)\ y_1'(t) + v_2'(t)\ y_2'(t)\right). \qquad (\ell_3) \end{aligned}$$

Method of Variation of Parameters

Equations for $v_1(t)$, $v_2(t)$:

$$\begin{aligned} v_1'(t)\, y_1(t) + v_2'(t)\, y_2(t) &= 0, \\ v_1'(t)\, y_1'(t) + v_2'(t)\, y_2'(t) &= \frac{f(t)}{a}. \end{aligned}$$

Set $\mathbf{wron}(t) \stackrel{def}{=} \mathbf{det}\begin{pmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{pmatrix}$, then

$$v_1'(t) = -\frac{f(t)\, y_2(t)}{a\, \mathbf{wron}(t)}, \quad v_2'(t) = \frac{f(t)\, y_1(t)}{a\, \mathbf{wron}(t)}.$$

Therefore,

$$v_1(t) = -\int^t \frac{f(\tau)\, y_2(\tau)}{a\, \mathbf{wron}(\tau)} d\,\tau, \quad v_2(t) = \int^t \frac{f(\tau)\, y_1(\tau)}{a\, \mathbf{wron}(\tau)} d\,\tau.$$

and $y_p(t) = v_1(t)\, y_1(t) + v_2(t)\, y_2(t)$ is solution to

$$a\, y'' + b\, y' + c\, y = f(t)$$

Method of Variation of Parameters: **Example (I)**

Find one PARTICULAR solution on $\left(-\frac{\pi}{2},\ \frac{\pi}{2}\right)$ for

$$y'' + y = \mathbf{tan}\,(t) \quad (\ell)$$

SOLUTION:

- Two linearly independent solutions for $y'' + y = 0$ are $y_1\,(t) = \mathbf{cos}\,(t)$, and $y_2\,(t) = \mathbf{sin}\,(t)$.
- Solution to (ℓ) : $y_p\,(t) = v_1\,(t)\ y_1\,(t) + v_2\,(t)\ y_2\,(t)$, with
- $\mathbf{wron}\,(t) \overset{def}{=} \mathbf{det}\begin{pmatrix} \mathbf{cos}\,(t) & \mathbf{sin}\,(t) \\ -\mathbf{sin}\,(t) & \mathbf{cos}\,(t) \end{pmatrix} = 1$, and

$$\begin{aligned} v_1\,(t) &= -\int^t \mathbf{tan}\,(\tau)\ \mathbf{sin}\,(\tau)\ d\,\tau = -\int^t \frac{1 - \mathbf{cos}^2\,(\tau)}{\mathbf{cos}\,(\tau)}\,d\,\tau \\ &= \mathbf{sin}\,(t) + \mathbf{ln}\left(\mathbf{tan}\left(\frac{\pi - 2\,t}{4}\right)\right) + c_1, \\ v_2\,(t) &= \int^t \mathbf{tan}\,(\tau)\ \mathbf{cos}\,(\tau)\ d\,\tau = -\mathbf{cos}\,(t) + c_2. \end{aligned}$$

Method of Variation of Parameters: **Example (I)**

Find one PARTICULAR solution on $\left(-\frac{\pi}{2},\ \frac{\pi}{2}\right)$ for

$$y'' + y = \tan(t) \quad (\ell)$$

SOLUTION: General solution

$$\begin{aligned} y(t) &= v_1(t)\cos(t) + v_2(t)\sin(t) \\ &= \cos(t)\ln\left(\tan\left(\frac{\pi - 2t}{4}\right)\right) + c_1\cos(t) + c_2\sin(t)\,. \end{aligned}$$

Setting $c_1 = c_2 = 0$ gives particular solution

$$y_p(t) = \cos(t)\ln\left(\tan\left(\frac{\pi - 2t}{4}\right)\right).$$

Method of Variation of Parameters: **Example** (II)

Find ALL solutions for variable-coefficient DE

$$t^2\, y'' - 4\, t\, y' + 6\, y = 4\, t^3, \quad t > 0, \quad (\ell)$$

SOLUTION:

- Lucky break: $y_1(t) = t^2$, $y_2(t) = t^3$ solutions for

$$t^2\, y'' - 4\, t\, y' + 6\, y = 0.$$

- Solution to (ℓ) : $y(t) = v_1(t)\ y_1(t) + v_2(t)\ y_2(t)$, with
- $\textbf{wron}(t) \stackrel{def}{=} \textbf{det}\begin{pmatrix} t^2 & t^3 \\ 2\,t & 3\,t^2 \end{pmatrix} = t^4$, and

$$\begin{aligned} v_1(t) &= -\int^t \frac{4\,\tau^3\,\tau^3}{\tau^2\,\tau^4}\, d\,\tau = -4\,t + \widehat{c}_1, \\ v_2(t) &= \int^t \frac{4\,\tau^3\,\tau^2}{\tau^2\,\tau^4}\, d\,\tau = 4\,\textbf{ln}(t) + \widehat{c}_2. \end{aligned}$$

General solution

$$y(t) = v_1(t)\ t^2 + v_2(t)\ t^3 = 4\,t^3\,\textbf{ln}(t) + c_1\,t^2 + c_2\,t^3.$$