The Analytic Theory of Heat, 1822

Fourier Analysis far more important than **Theory of Heat**

§10.3 Fourier series

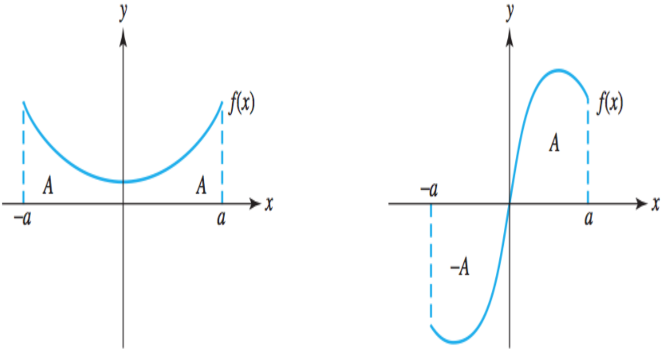
- function $f(x)$ is **periodic with period** T if $f(x) = f(x + T)$ for all x.

§10.3 Fourier series

- function $f(x)$ is **periodic with period** T if $f(x) = f(x+T)$ for all x.
- function $f(x)$ is **even** $f(-x) = f(x)$ for all x (Fig. (a)). $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$.

§10.3 Fourier series

- function $f(x)$ is **periodic with period** T if $f(x) = f(x+T)$ for all x.
- function $f(x)$ is **even** $f(-x) = f(x)$ for all x (Fig. (a)).
  $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$.
- function $f(x)$ is **odd** $f(-x) = -f(x)$ for all (Fig. (b)).
  $\int_{-a}^{a} f(x)\,dx = 0$



§10.3 Fourier series: **Examples**

- function $f(x) = \mathbf{sin}(3x)$ is **periodic with period** $\frac{2}{3}\pi$:

$$f\left(x + \frac{2}{3}\pi\right) = \mathbf{sin}\left(3\left(x + \frac{2}{3}\pi\right)\right) = \mathbf{sin}(3x) = f(x).$$

§10.3 Fourier series: **Examples**

- function $f(x) = \sin(3x)$ is **periodic with period** $\frac{2}{3}\pi$:

$$f\left(x + \frac{2}{3}\pi\right) = \sin\left(3\left(x + \frac{2}{3}\pi\right)\right) = \sin(3x) = f(x)\,.$$

- function $f(x) = \sqrt{1 + x^2}$ is **even**

$$f(-x) = \sqrt{1 + (-x)^2} = \sqrt{1 + x^2} = f(x)\,.$$

§10.3 Fourier series: **Examples**

- function $f(x) = \mathbf{sin}(3x)$ is **periodic with period** $\frac{2}{3}\pi$:

$$f\left(x + \frac{2}{3}\pi\right) = \mathbf{sin}\left(3\left(x + \frac{2}{3}\pi\right)\right) = \mathbf{sin}(3x) = f(x).$$

- function $f(x) = \sqrt{1 + x^2}$ is **even**

$$f(-x) = \sqrt{1 + (-x)^2} = \sqrt{1 + x^2} = f(x).$$

- function $f(x) = \mathbf{sin}(3x)$ is **odd**

$$f(-x) = \mathbf{sin}(3(-x)) = -\mathbf{sin}(3x) = -f(x).$$

Inner Product Space: **Review**

Let V be a vector space. **inner product** is a function

$$V \times V \longmapsto \mathcal{R} : \ < \mathbf{u}, \mathbf{v} > \ \in \mathcal{R} \quad \text{for any} \quad \mathbf{u}, \mathbf{v} \in V$$

that satisfies axioms below for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and $c \in \mathcal{R}$:

1. $< \mathbf{u}, \mathbf{v} >=< \mathbf{v}, \mathbf{u} >$. (Symmetry with respect to $\mathbf{u}$ and $\mathbf{v}$)

Inner Product Space: **Review**

Let V be a vector space. **inner product** is a function

$$V \times V \longmapsto \mathcal{R} : \ < \mathbf{u}, \mathbf{v} > \ \in \mathcal{R} \quad \text{for any} \quad \mathbf{u}, \mathbf{v} \in V$$

that satisfies axioms below for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and $c \in \mathcal{R}$:

1. $< \mathbf{u}, \mathbf{v} >=< \mathbf{v}, \mathbf{u} >$. (Symmetry with respect to **u** and **v**)
2. $< \mathbf{u} + \mathbf{w}, \mathbf{v} >=< \mathbf{u}, \mathbf{v} > + < \mathbf{w}, \mathbf{v} >$.
3. $< c\,\mathbf{u}, \mathbf{v} >= c \ < \mathbf{u}, \mathbf{v} >$. (Linear transformation in **u**)

Inner Product Space: **Review**

Let V be a vector space. **inner product** is a function

$$V \times V \longmapsto \mathcal{R} : \; < \mathbf{u}, \mathbf{v} > \; \in \mathcal{R} \quad \text{for any} \quad \mathbf{u}, \mathbf{v} \in V$$

that satisfies axioms below for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and $c \in \mathcal{R}$:

1. $< \mathbf{u}, \mathbf{v} > = < \mathbf{v}, \mathbf{u} >$. (Symmetry with respect to **u** and **v**)
2. $< \mathbf{u} + \mathbf{w}, \mathbf{v} > = < \mathbf{u}, \mathbf{v} > + < \mathbf{w}, \mathbf{v} >$.
3. $< c\,\mathbf{u}, \mathbf{v} > = c \; < \mathbf{u}, \mathbf{v} >$. (Linear transformation in **u**)
4. $< \mathbf{u}, \mathbf{u} > \; \geq 0$, and $< \mathbf{u}, \mathbf{u} > \; = 0$ if and only if $\mathbf{u} = \mathbf{0}$.

Inner Product Space: **Review**

Let V be a vector space. **inner product** is a function

$$V \times V \longmapsto \mathcal{R} : \ < \mathbf{u}, \mathbf{v} > \ \in \mathcal{R} \quad \text{for any} \quad \mathbf{u}, \mathbf{v} \in V$$

that satisfies axioms below for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and $c \in \mathcal{R}$:

1. $< \mathbf{u}, \mathbf{v} > = < \mathbf{v}, \mathbf{u} >$. (<u>Symmetry with respect to **u** and **v**</u>)
2. $< \mathbf{u} + \mathbf{w}, \mathbf{v} > = < \mathbf{u}, \mathbf{v} > + < \mathbf{w}, \mathbf{v} >$.
3. $< c\,\mathbf{u}, \mathbf{v} > = c \ < \mathbf{u}, \mathbf{v} >$. (<u>Linear transformation in **u**</u>)
4. $< \mathbf{u}, \mathbf{u} > \ \geq 0$, and $< \mathbf{u}, \mathbf{u} > \ = 0$ if and only if $\mathbf{u} = \mathbf{0}$.

Example: For any $f(x), g(x) \in C[-L, L]$, then

$$< f, g > \stackrel{def}{=} \int_{-L}^{L} f(x)\, g(x)\, dx \quad \text{is an inner product on } C[-L, L].$$

- **length** (or **norm**) of $\mathbf{u}$ (denoted $\|\mathbf{u}\|$) $\stackrel{def}{=} \sqrt{<\mathbf{u}, \mathbf{u}>}$

- **length** (or **norm**) of $\mathbf{u}$ (denoted $\|\mathbf{u}\|$) $\stackrel{def}{=} \sqrt{< \mathbf{u}, \mathbf{u} >}$
- **distance** between $\mathbf{u}$ and $\mathbf{v}$ (denoted $\mathbf{dist}\,(\mathbf{u}, \mathbf{v})$) $\stackrel{def}{=} \|\mathbf{u} - \mathbf{v}\|$

- **length** (or **norm**) of $\mathbf{u}$ (denoted $\|\mathbf{u}\|$) $\stackrel{def}{=} \sqrt{< \mathbf{u}, \mathbf{u} >}$
- **distance** between $\mathbf{u}$ and $\mathbf{v}$ (denoted $\mathbf{dist}\,(\mathbf{u}, \mathbf{v})$) $\stackrel{def}{=} \|\mathbf{u} - \mathbf{v}\|$
- $\mathbf{u}$ and $\mathbf{v}$ are **orthogonal** if $< \mathbf{u}, \mathbf{v} > = 0$

- **length** (or **norm**) of $\mathbf{u}$ (denoted $\|\mathbf{u}\|$) $\stackrel{def}{=} \sqrt{< \mathbf{u}, \mathbf{u} >}$
- **distance** between $\mathbf{u}$ and $\mathbf{v}$ (denoted $\mathbf{dist}\,(\mathbf{u}, \mathbf{v})$) $\stackrel{def}{=} \|\mathbf{u} - \mathbf{v}\|$
- $\mathbf{u}$ and $\mathbf{v}$ are **orthogonal** if $< \mathbf{u}, \mathbf{v} > = 0$

- **u** and **v** are **orthogonal** if $<\mathbf{u}, \mathbf{v}> = 0$

EX: For integer $n > 0$, show that $f(x) \equiv 1$ and $g(x) = \mathbf{cos}\left(\frac{n\pi x}{L}\right)$ are **orthogonal** with respect to

$$\text{inner product} \quad < f, g > \stackrel{def}{=} \int_{-L}^{L} f(x)\, g(x)\, dx.$$

PROOF:

$$< f, g > \;=\; \int_{-L}^{L} \mathbf{cos}\left(\frac{n\pi x}{L}\right) dx = 0.$$

- **u** and **v** are **orthogonal** if $< \mathbf{u}, \mathbf{v} > = 0$

EX: For integers $m, n \geq 0$, show that $f(x) = \sin\left(\frac{m\pi x}{L}\right)$ and $g(x) = \cos\left(\frac{n\pi x}{L}\right)$ are **orthogonal** with respect to

$$\text{inner product} \quad < f, g > \stackrel{def}{=} \int_{-L}^{L} f(x)\, g(x)\, dx.$$

PROOF:

$$\begin{aligned} f(x)\, g(x) &= \sin\left(\frac{m\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right) \\ &= \frac{1}{2}\left(\sin\left(\frac{(m-n)\pi x}{L}\right) + \sin\left(\frac{(m+n)\pi x}{L}\right)\right), \end{aligned}$$

which is sum of two <u>odd</u> functions, thus $\int_{-L}^{L} f(x)\, g(x)\, dx = 0$.

- **u** and **v** are **orthogonal** if $< \mathbf{u}, \mathbf{v} > = 0$

EX: For integers $m, n \geq 0$, show that $f(x) = \sin\left(\frac{m\pi x}{L}\right)$ and $g(x) = \cos\left(\frac{n\pi x}{L}\right)$ are **orthogonal** with respect to

$$\text{inner product} \quad < f, g > \stackrel{def}{=} \int_{-L}^{L} f(x)\, g(x)\, dx.$$

PROOF:

$$\begin{aligned} f(x)\, g(x) &= \sin\left(\frac{m\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right) \\ &= \frac{1}{2}\left(\sin\left(\frac{(m-n)\pi x}{L}\right) + \sin\left(\frac{(m+n)\pi x}{L}\right)\right), \end{aligned}$$

which is sum of two <u>odd</u> functions, thus $\int_{-L}^{L} f(x)\, g(x)\, dx = 0$.

Setting $n = 0$, $f(x) = \sin\left(\frac{m\pi x}{L}\right)$ and $g(x) \equiv 1$ are **orthogonal** for integer $m > 0$.

- **u** and **v** are **orthogonal** if $<\mathbf{u}, \mathbf{v}> \; = 0$

EX: For integers $m, n > 0$ with $m \neq n$, show that $f(x) = \sin\left(\frac{m\pi x}{L}\right)$ and $g(x) = \sin\left(\frac{n\pi x}{L}\right)$ are **orthogonal**

with respect to inner product $\quad < f, g > \stackrel{def}{=} \int_{-L}^{L} f(x)\, g(x)\, dx.$

PROOF:

$$\begin{aligned} f(x)\, g(x) &= \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) \\ &= \frac{1}{2}\left(\cos\left(\frac{(m-n)\pi x}{L}\right) - \cos\left(\frac{(m+n)\pi x}{L}\right)\right), \end{aligned}$$

therefore

$$\begin{aligned} \int_{-L}^{L} f(x)\, g(x)\, dx &= \frac{1}{2}\int_{-L}^{L} \cos\left(\frac{(m-n)\pi x}{L}\right) dx \\ &\quad -\frac{1}{2}\int_{-L}^{L} \cos\left(\frac{(m+n)\pi x}{L}\right) dx = 0. \end{aligned}$$

- **u** and **v** are **orthogonal** if $<\mathbf{u}, \mathbf{v}> = 0$

EX: For integers $m, n > 0$ with $m \neq n$, show that $f(x) = \mathbf{cos}\left(\frac{m\pi x}{L}\right)$ and $g(x) = \mathbf{cos}\left(\frac{n\pi x}{L}\right)$ are **orthogonal**

with respect to inner product $\quad < f, g > \stackrel{def}{=} \int_{-L}^{L} f(x)\, g(x)\, dx.$

PROOF:

$$
\begin{aligned}
f(x)\, g(x) &= \cos\left(\frac{m\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right) \\
&= \frac{1}{2}\left(\cos\left(\frac{(m-n)\pi x}{L}\right) + \cos\left(\frac{(m+n)\pi x}{L}\right)\right),
\end{aligned}
$$

therefore

$$
\begin{aligned}
\int_{-L}^{L} f(x)\, g(x)\, dx &= \frac{1}{2}\int_{-L}^{L} \cos\left(\frac{(m-n)\pi x}{L}\right) dx \\
&\quad + \frac{1}{2}\int_{-L}^{L} \cos\left(\frac{(m+n)\pi x}{L}\right) dx = 0.
\end{aligned}
$$

Orthogonal functions summary

$\left\{1, \sin\left(\frac{\pi x}{L}\right), \cos\left(\frac{\pi x}{L}\right), \cdots, \sin\left(\frac{n\pi x}{L}\right), \cos\left(\frac{n\pi x}{L}\right), \cdots, \right\}$
mutually **orthogonal** with respect to

$$\text{inner product} \quad < f, g > \stackrel{def}{=} \int_{-L}^{L} f(x)\, g(x)\, dx.$$

- $\left\{1, \sin\left(\frac{\pi x}{L}\right), \cos\left(\frac{\pi x}{L}\right), \cdots, \sin\left(\frac{n\pi x}{L}\right), \cos\left(\frac{n\pi x}{L}\right), \cdots, \right\}$
is a set of infinitely many linearly independent functions.
- Inner Product Space $C[-L, L]$ is NOT finite-dimensional.

Orthogonal Sinusoids

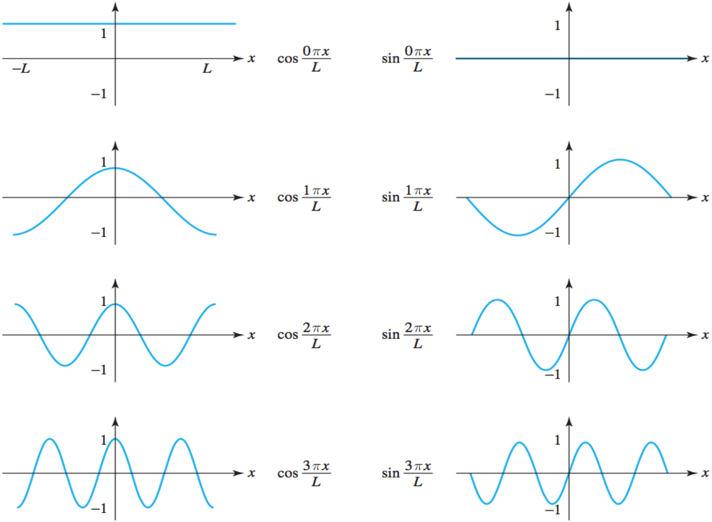


Figure 10.5 The sinusoids

Lengths of orthogonal functions

- **length** (or **norm**) of $\mathbf{u}$ (denoted $\|\mathbf{u}\|$) $\stackrel{def}{=} \sqrt{<\mathbf{u}, \mathbf{u}>}$

Lengths of orthogonal functions

- **length** (or **norm**) of $\mathbf{u}$ (denoted $\|\mathbf{u}\|$) $\stackrel{def}{=} \sqrt{< \mathbf{u}, \mathbf{u} >}$
- **length** of $f_0(x) \equiv 1$

$$\|f_0\| = \sqrt{\int_{-L}^{L} 1^2 \, dx} = \sqrt{2L}.$$

Lengths of orthogonal functions

- **length** (or **norm**) of $\mathbf{u}$ (denoted $\|\mathbf{u}\|$) $\stackrel{def}{=} \sqrt{< \mathbf{u}, \mathbf{u} >}$
- **length** of $f_0(x) \equiv 1$

$$\|f_0\| = \sqrt{\int_{-L}^{L} 1^2 \, dx} = \sqrt{2\,L}.$$

- **length** of $f_n(x) \stackrel{def}{=} \mathbf{cos}\left(\frac{n\,\pi\,x}{L}\right)$ for $n > 0$:

$$\|f_n\| = \sqrt{\int_{-L}^{L} \mathbf{cos}^2\left(\frac{n\,\pi\,x}{L}\right) \, dx} = \sqrt{L}.$$

Lengths of orthogonal functions

- **length** (or **norm**) of $\mathbf{u}$ (denoted $\|\mathbf{u}\|$) $\stackrel{def}{=} \sqrt{<\mathbf{u}, \mathbf{u}>}$
- **length** of $f_0(x) \equiv 1$

$$\|f_0\| = \sqrt{\int_{-L}^{L} 1^2\, dx} = \sqrt{2\,L}.$$

- **length** of $f_n(x) \stackrel{def}{=} \mathbf{cos}\left(\frac{n\,\pi\,x}{L}\right)$ for $n > 0$:

$$\|f_n\| = \sqrt{\int_{-L}^{L} \mathbf{cos}^2\left(\frac{n\,\pi\,x}{L}\right)\, dx} = \sqrt{L}.$$

- **length** of $g_n(x) \stackrel{def}{=} \mathbf{sin}\left(\frac{n\,\pi\,x}{L}\right)$ for $n > 0$:

$$\|g_n\| = \sqrt{\int_{-L}^{L} \mathbf{sin}^2\left(\frac{n\,\pi\,x}{L}\right)\, dx} = \sqrt{L}.$$

Best Approximation (Review)

Let W be a subspace of inner product space V, with orthogonal basis $\{\mathbf{u}_1, \cdots, \mathbf{u}_p\}$.

Best Approximation (Review)

Let W be a subspace of inner product space V, with orthogonal basis $\{\mathbf{u}_1, \cdots, \mathbf{u}_p\}$.

- DEFINITION: For any vector $\mathbf{y} \in V$, **orthogonal projection**

$$\mathbf{proj}_W \mathbf{y} \stackrel{def}{=} \widehat{\mathbf{y}} = \frac{< \mathbf{y}, \mathbf{u}_1 >}{< \mathbf{u}_1, \mathbf{u}_1 >}\mathbf{u}_1 + \cdots + \frac{< \mathbf{y}, \mathbf{u}_p >}{< \mathbf{u}_p, \mathbf{u}_p >}\mathbf{u}_p$$

Best Approximation (Review)

Let W be a subspace of inner product space V, with orthogonal basis $\{\mathbf{u}_1, \cdots, \mathbf{u}_p\}$.

- DEFINITION: For any vector $\mathbf{y} \in V$, **orthogonal projection**

$$\mathbf{proj}_W \mathbf{y} \stackrel{def}{=} \widehat{\mathbf{y}} = \frac{<\mathbf{y}, \mathbf{u}_1>}{<\mathbf{u}_1, \mathbf{u}_1>}\mathbf{u}_1 + \cdots + \frac{<\mathbf{y}, \mathbf{u}_p>}{<\mathbf{u}_p, \mathbf{u}_p>}\mathbf{u}_p$$

- **Pythagorean Thm**:

$$\|\mathbf{y}\|^2 = \|\mathbf{y} - \widehat{\mathbf{y}}\|^2 + \|\widehat{\mathbf{y}}\|^2, \quad \|\mathbf{y} - \mathbf{v}\| \geq \|\mathbf{y} - \widehat{\mathbf{y}}\|, \quad \text{for all } \mathbf{v} \in W.$$

Best Approximation (Review)

Let W be a subspace of inner product space V, with orthogonal basis $\{\mathbf{u}_1, \cdots, \mathbf{u}_p\}$.

- DEFINITION: For any vector $\mathbf{y} \in V$, **orthogonal projection**

$$\mathbf{proj}_W \mathbf{y} \stackrel{def}{=} \widehat{\mathbf{y}} = \frac{< \mathbf{y}, \mathbf{u}_1 >}{< \mathbf{u}_1, \mathbf{u}_1 >}\mathbf{u}_1 + \cdots + \frac{< \mathbf{y}, \mathbf{u}_p >}{< \mathbf{u}_p, \mathbf{u}_p >}\mathbf{u}_p$$

- **Pythagorean Thm**:

$$\|\mathbf{y}\|^2 = \|\mathbf{y} - \widehat{\mathbf{y}}\|^2 + \|\widehat{\mathbf{y}}\|^2, \quad \|\mathbf{y} - \mathbf{v}\| \geq \|\mathbf{y} - \widehat{\mathbf{y}}\|, \quad \text{for all } \mathbf{v} \in W.$$

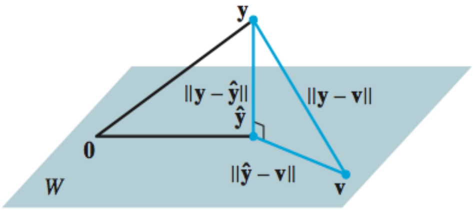


FIGURE 4 The orthogonal projection of **y** onto *W* is the closest point in *W* to **y**.

Orthogonal projection with $< f, g > \stackrel{def}{=} \int_{-L}^{L} f(x)\, g(x)\, dx$

For $f(x) \in C[-L, L]$, its **orthogonal projection** $\mathcal{S}_N(x)$ onto

$$W_N \stackrel{def}{=} \textbf{Span}\left\{1, \textbf{sin}\left(\frac{\pi x}{L}\right), \textbf{cos}\left(\frac{\pi x}{L}\right), \cdots, \textbf{sin}\left(\frac{N\pi x}{L}\right), \textbf{cos}\left(\frac{N\pi x}{L}\right)\right\}:$$

$$\mathcal{S}_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N} a_n \textbf{cos}\left(\frac{n\pi x}{L}\right) + b_n \textbf{sin}\left(\frac{n\pi x}{L}\right), \quad \text{where}$$

$$a_n = \frac{1}{L} < f(x), \textbf{cos}\left(\frac{n\pi x}{L}\right) >= \frac{1}{L}\int_{-L}^{L} f(x)\, \textbf{cos}\left(\frac{n\pi x}{L}\right) dx, \; 0 \le n \le N$$

$$b_n = \frac{1}{L} < f(x), \textbf{sin}\left(\frac{n\pi x}{L}\right) >= \frac{1}{L}\int_{-L}^{L} f(x)\, \textbf{sin}\left(\frac{n\pi x}{L}\right) dx, \; 1 \le n \le N$$

Orthogonal projection with $< f, g > \stackrel{def}{=} \int_{-L}^{L} f(x)\, g(x)\, dx$

For $f(x) \in C[-L, L]$, its **orthogonal projection** $\mathcal{S}_N(x)$ onto

$$W_N \stackrel{def}{=} \textbf{Span}\left\{1,\ \textbf{sin}\left(\frac{\pi x}{L}\right), \textbf{cos}\left(\frac{\pi x}{L}\right), \cdots, \textbf{sin}\left(\frac{N\pi x}{L}\right), \textbf{cos}\left(\frac{N\pi x}{L}\right)\right\}:$$

$$\mathcal{S}_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N} a_n \textbf{cos}\left(\frac{n\pi x}{L}\right) + b_n \textbf{sin}\left(\frac{n\pi x}{L}\right), \quad \text{where}$$

$$a_n = \frac{1}{L} < f(x), \textbf{cos}\left(\frac{n\pi x}{L}\right) >= \frac{1}{L}\int_{-L}^{L} f(x)\, \textbf{cos}\left(\frac{n\pi x}{L}\right) dx,\ 0 \le n \le N$$

$$b_n = \frac{1}{L} < f(x), \textbf{sin}\left(\frac{n\pi x}{L}\right) >= \frac{1}{L}\int_{-L}^{L} f(x)\, \textbf{sin}\left(\frac{n\pi x}{L}\right) dx,\ 1 \le n \le N$$

Big Hope $f(x) = \textbf{lim}_{N\to\infty}\mathcal{S}_N(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \textbf{cos}\left(\frac{n\pi x}{L}\right) + b_n \textbf{sin}\left(\frac{n\pi x}{L}\right)$

EX 1: Compute Fourier Series for $f(x) = |x| \in C[-1,1]$

$$\text{In} \quad \mathcal{S}_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N} a_n \cos\left(\frac{n\,\pi\,x}{L}\right) + b_n \sin\left(\frac{n\,\pi\,x}{L}\right), \quad \text{where}$$

$$a_0 = \int_{-1}^{1} |x| \; dx = 1, \quad \text{and for } n \geq 1,$$

$$a_n = \int_{-1}^{1} |x| \, \cos(n\,\pi\,x)\, dx = 2 \int_{0}^{1} x \cos(n\,\pi\,x)\, dx = \frac{2}{\pi^2\, n^2}\left((-1)^n - 1\right)$$

$$b_n = \int_{-1}^{1} |x| \, \sin(n\,\pi\,x)\, dx = 0.$$

EX 1: Compute Fourier Series for $f(x) = |x| \in C[-1,1]$

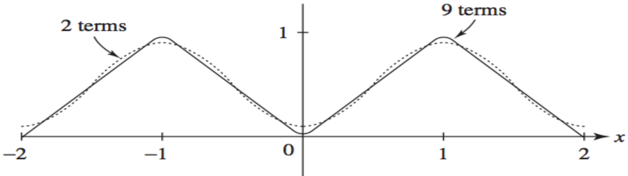
In $$\mathcal{S}_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right),$$ where

$$a_0 = \int_{-1}^{1} |x|\, dx = 1, \quad \text{and for } n \geq 1,$$

$$a_n = \int_{-1}^{1} |x| \cos(n\pi x)\, dx = 2\int_{0}^{1} x\cos(n\pi x)\, dx = \frac{2}{\pi^2 n^2}\left((-1)^n - 1\right)$$

$$b_n = \int_{-1}^{1} |x| \sin(n\pi x)\, dx = 0.$$

Therefore $$|x| \sim \frac{1}{2} - \frac{4}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}\cos\left((2n-1)\pi x\right).$$



EX 2: Compute Fourier Series for $f(x) = \begin{cases} 0, & -\pi < x < 0, \\ x & 0 < x < \pi. \end{cases}$

$$\text{In} \quad \mathcal{S}_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right), \quad \text{where}$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx = \frac{1}{\pi}\int_{0}^{\pi} x\, dx = \frac{\pi}{2}, \quad \text{and for } n \geq 1,$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos(n\pi x)\, dx = \frac{1}{\pi}\int_{0}^{\pi} x \cos(n\pi x)\, dx = \frac{2}{\pi n^2}\left((-1)^n - 1\right)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin(n\pi x)\, dx = \frac{1}{\pi}\int_{0}^{\pi} x \sin(n\pi x)\, dx = \frac{1}{n}(-1)^{n+1}$$

EX 2: Compute Fourier Series for $f(x) = \begin{cases} 0, & -\pi < x < 0, \\ x & 0 < x < \pi. \end{cases}$

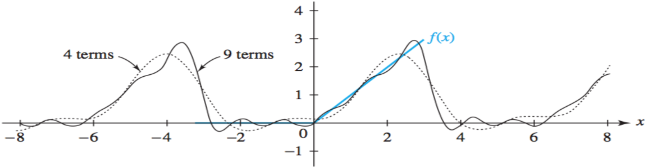
In $\mathcal{S}_N(x) = \dfrac{a_0}{2} + \displaystyle\sum_{n=1}^{N} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)$, where

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx = \frac{1}{\pi}\int_0^{\pi} x\, dx = \frac{\pi}{2}, \quad \text{and for } n \geq 1,$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos(n\pi x)\, dx = \frac{1}{\pi}\int_0^{\pi} x \cos(n\pi x)\, dx = \frac{2}{\pi n^2}\left((-1)^n - 1\right)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin(n\pi x)\, dx = \frac{1}{\pi}\int_0^{\pi} x \sin(n\pi x)\, dx = \frac{1}{n}(-1)^{n+1}$$

So $f(x) \sim \dfrac{\pi}{4} - \dfrac{2}{\pi}\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos((2n-1)x) - \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin(nx)$.



Pointwise Convergence of Fourier Series, **Thm. 2**

If f and f' are piecewise continuous on $[-L, L]$, then for any $x \in (-L, L)$, the Fourier series

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}\left\{a_n \cos\left(\frac{n\,\pi\,x}{L}\right) + b_n \sin\left(\frac{n\,\pi\,x}{L}\right)\right\} = \frac{1}{2}\left(f\left(x^{+}\right) + f\left(x^{-}\right)\right).$$

For $x = \pm L$, the series converges to $\frac{1}{2}\left(f\left(-L^{+}\right) + f\left(L^{-}\right)\right)$.

Fourier Series Calculus

If f and f' are continuous on $[-L, L]$ so that for any $x \in (-L, L)$, the Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left\{ a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right\}.$$

Then

$$\begin{aligned} f'(x) &= \sum_{n=1}^{\infty} \frac{\pi n}{L} \left\{ -a_n \sin\left(\frac{n\pi x}{L}\right) + b_n \cos\left(\frac{n\pi x}{L}\right) \right\}, \\ \int_{-L}^{x} f(t)\, dt &= \int_{-L}^{x} \frac{a_0}{2}\, dt + \sum_{n=1}^{\infty} \int_{-L}^{x} \left\{ a_n \cos\left(\frac{n\pi t}{L}\right) + b_n \sin\left(\frac{n\pi t}{L}\right) \right\} dt. \end{aligned}$$

Fourier Series with Inner Prod. $<f, g> \stackrel{def}{=} \int_{-L}^{L} f(x)\, g(x)\, dx$

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right), \quad \text{where}$$

$$a_n = \frac{1}{L} < f(x), \cos\left(\frac{n\pi x}{L}\right) >= \frac{1}{L}\int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx, \;\; n \geq 0$$

$$b_n = \frac{1}{L} < f(x), \sin\left(\frac{n\pi x}{L}\right) >= \frac{1}{L}\int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx, n \geq 1$$

Fourier Series with Inner Prod. $<f,g> \overset{def}{=} \int_{-L}^{L} f(x)\, g(x)\, dx$

$$f(x) \;\sim\; \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right), \quad \text{where}$$

$$a_n = \frac{1}{L} < f(x), \cos\left(\frac{n\pi x}{L}\right) >= \frac{1}{L}\int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx, \;\; n \geq 0$$

$$b_n = \frac{1}{L} < f(x), \sin\left(\frac{n\pi x}{L}\right) >= \frac{1}{L}\int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx, n \geq 1$$

Given function $f(x)$ on $(0, L)$, then

- **even extension:** Define $f(x) = f(-x)$ on $(-L, 0)$,

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx = \frac{2}{L}\int_{0}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right), \;\; n \geq 0$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx = 0, n \geq 1,$$

$$f(x) \;\sim\; \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right).$$

Fourier Series with Inner Prod. $<f, g> \stackrel{def}{=} \int_{-L}^{L} f(x)\, g(x)\, dx$

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right), \quad \text{where}$$

$$a_n = \frac{1}{L} < f(x), \cos\left(\frac{n\pi x}{L}\right) >= \frac{1}{L}\int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx, \;\; n \geq 0$$

$$b_n = \frac{1}{L} < f(x), \sin\left(\frac{n\pi x}{L}\right) >= \frac{1}{L}\int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx, n \geq 1$$

Fourier Series with Inner Prod. $<f,g> \overset{def}{=} \int_{-L}^{L} f(x)\, g(x)\, dx$

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right), \quad \text{where}$$

$$a_n = \frac{1}{L} < f(x), \cos\left(\frac{n\pi x}{L}\right) >= \frac{1}{L}\int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx, \ n \geq 0$$

$$b_n = \frac{1}{L} < f(x), \sin\left(\frac{n\pi x}{L}\right) >= \frac{1}{L}\int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx, n \geq 1$$

Given function $f(x)$ on $(0, L)$, then

- **odd extension:** Define $f(x) = -f(-x)$ on $(-L, 0)$,

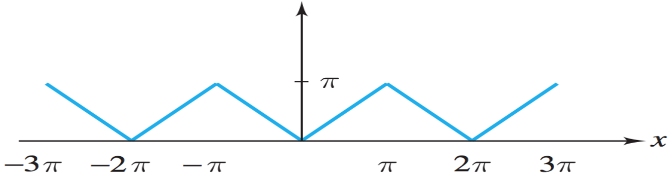
$$a_n = \frac{1}{L}\int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx = 0, \ n \geq 0$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx = \frac{2}{L}\int_{0}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx, n \geq 1,$$

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right).$$

Example: Even Fourier Series for $f(x) = x$ on $(0, \pi)$

even extension: $$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) \sim f(x) = \begin{cases} x, & \text{if } x > 0, \\ -x, & \text{if } x < 0. \end{cases}$$



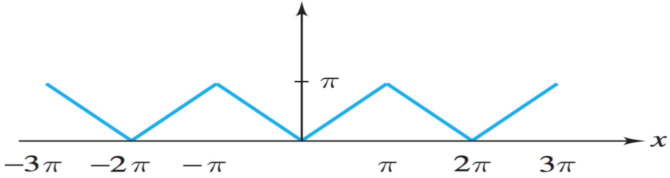
(c) Even 2π periodic

$$a_n = \frac{2}{\pi}\int_0^{\pi} x \cos(n x)\, dx = \begin{cases} \pi, & \text{for } n = 0, \\ 0, & \text{for } n > 0 \text{ even}, \\ -\frac{4}{\pi n^2}, & \text{for } n \text{ odd}. \end{cases}$$

$$x = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=0}^{\infty} \frac{\cos((2n+1)\, x)}{(2n+1)^2} \quad \text{for } x \in [0, \pi].$$

Example: Even Fourier Series for $f(x) = x$ on $(0, \pi)$

even extension: $$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) \sim f(x) = \begin{cases} x, & \text{if } x > 0, \\ -x, & \text{if } x < 0. \end{cases}$$



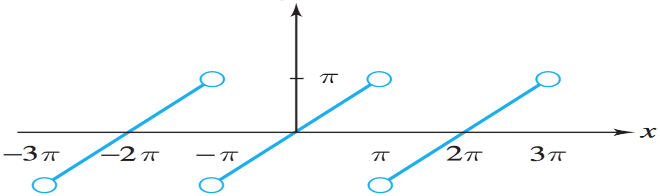
(c) Even 2π periodic

$$a_n = \frac{2}{\pi}\int_0^{\pi} x \cos(nx)\, dx = \begin{cases} \pi, & \text{for } n = 0, \\ 0, & \text{for } n > 0 \text{ even}, \\ -\frac{4}{\pi n^2}, & \text{for } n \text{ odd}. \end{cases}$$

$$x = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=0}^{\infty} \frac{\cos((2n+1)\, x)}{(2n+1)^2} \quad \text{for } x \in [0, \pi]. \implies \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}.$$

Example: Odd Fourier Series for $f(x) = x$ on $(0, \pi)$

odd extension: $\sum^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \sim f(x) = \begin{cases} x, & \text{if } x > 0, \\ x, & \text{if } x < 0. \end{cases}$



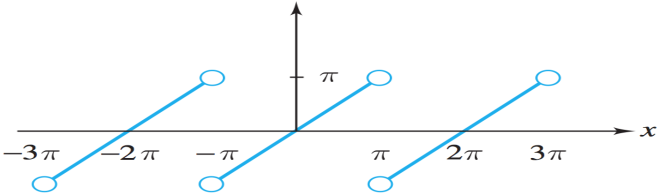
(b) Odd 2π periodic

$$b_n = \frac{2}{\pi} \int_0^{\pi} x \sin(nx)\, dx = -\frac{2}{n} (-1)^{n+1}.$$

$$x = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(nx) \quad \text{for } x \in [0, \pi].$$

Example: Odd Fourier Series for $f(x) = x$ on $(0, \pi)$

odd extension: $\displaystyle \sum^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \sim f(x) = \begin{cases} x, & \text{if } x > 0, \\ x, & \text{if } x < 0. \end{cases}$



(b) Odd 2π periodic

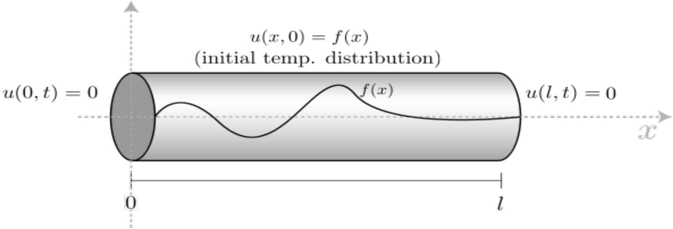
$$b_n = \frac{2}{\pi}\int_0^{\pi} x \sin(nx)\, dx = -\frac{2}{n}(-1)^{n+1}.$$

$$x = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(nx) \quad \text{for } x \in [0, \pi].$$

cf. **even extension:** $\displaystyle x = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=0}^{\infty} \frac{\cos((2n+1)x)}{(2n+1)^2}$ for $x \in [0, \pi]$.

§10.5 Heat conduction model (Fourier, 1822)

$u = u(x,t)$ is temperature at position x at time t



Governing partial differential equation

$$\frac{\partial u}{\partial t} = \beta, \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L, \quad t > 0.$$

- Assume initial condition $u(x,0) = f(x) \quad \forall x \in [0, L]$, with a given function f.
- the boundary conditions $u(0,t) = 0 = u(L,t) \quad \forall t > 0$.

$$
\begin{aligned}
\frac{\partial u}{\partial t} &= \beta \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L, \quad t > 0. \quad (\ell) \\
u(x,0) &= f(x) \quad \forall\, x \in [0, L]; \qquad u(0,t) = 0 = u(L,t) \quad \forall\, t > 0.
\end{aligned}
$$

$$\begin{aligned} \frac{\partial u}{\partial t} &= \beta \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L, \quad t > 0. \quad (\ell) \\ u(x,0) &= f(x) \quad \forall\, x \in [0, L]; \qquad u(0,t) = 0 = u(L,t) \quad \forall\, t > 0. \end{aligned}$$

SEPARATION OF VARIABLES: First try to find a solution in form

$$u(x,t) = \mathbf{X}(x)\, \mathbf{T}(t) \quad \text{that}$$

- satisfies boundary conditions $\mathbf{X}(0) = \mathbf{X}(L) = 0$,
- is NOT identically zero.

$$\frac{\partial u}{\partial t} = \beta \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L, \quad t > 0. \quad (\ell)$$
$$u(x,0) = f(x) \quad \forall\, x \in [0,L]; \qquad u(0,t) = 0 = u(L,t) \quad \forall\, t > 0.$$

SEPARATION OF VARIABLES: First try to find a solution in form

$$u(x,t) = \mathbf{X}(x)\, \mathbf{T}(t) \quad \text{that}$$

- satisfies boundary conditions $\mathbf{X}(0) = \mathbf{X}(L) = 0$,
- is NOT identically zero.

From $\dfrac{\partial u}{\partial t} = \mathbf{X}(x)\, \mathbf{T}'(t)$, $\dfrac{\partial^2 u}{\partial x^2} = \mathbf{X}''(x)\, \mathbf{T}(t)$

$$\frac{\partial u}{\partial t} = \beta \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L, \quad t > 0. \quad (\ell)$$

$$u(x,0) = f(x) \quad \forall\, x \in [0, L]; \qquad u(0,t) = 0 = u(L,t) \quad \forall\, t > 0.$$

SEPARATION OF VARIABLES: First try to find a solution in form

$$u(x,t) = \mathbf{X}(x)\ \mathbf{T}(t) \quad \text{that}$$

- satisfies boundary conditions $\mathbf{X}(0) = \mathbf{X}(L) = 0$,
- is NOT identically zero.

From $\dfrac{\partial u}{\partial t} = \mathbf{X}(x)\ \mathbf{T}'(t)$, $\dfrac{\partial^2 u}{\partial x^2} = \mathbf{X}''(x)\ \mathbf{T}(t)$

$$\implies \mathbf{X}(x)\ \mathbf{T}'(t) = \beta\, \mathbf{X}''(x)\ \mathbf{T}(t)$$

$$\begin{aligned} \frac{\partial u}{\partial t} &= \beta \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L, \quad t > 0. \quad (\ell) \\ u(x,0) &= f(x) \quad \forall\, x \in [0, L]; \qquad u(0,t) = 0 = u(L,t) \quad \forall\, t > 0. \end{aligned}$$

SEPARATION OF VARIABLES: First try to find a solution in form

$$u(x,t) = \mathbf{X}(x)\ \mathbf{T}(t) \quad \text{that}$$

- satisfies boundary conditions $\mathbf{X}(0) = \mathbf{X}(L) = 0$,
- is NOT identically zero.

$$\text{From } \frac{\partial u}{\partial t} = \mathbf{X}(x)\ \mathbf{T}'(t), \quad \frac{\partial^2 u}{\partial x^2} = \mathbf{X}''(x)\ \mathbf{T}(t)$$

$$\Longrightarrow \quad \mathbf{X}(x)\ \mathbf{T}'(t) = \beta\, \mathbf{X}''(x)\ \mathbf{T}(t) \quad \Longrightarrow \quad \frac{\mathbf{X}''(x)}{\mathbf{X}(x)} = \frac{\mathbf{T}'(t)}{\beta\, \mathbf{T}(t)} \stackrel{def}{=} -\lambda.$$

λ: neither function of x nor t, therefore must be certain constant

Determine values of λ: trivial cases

Now find $\mathbf{X}(x)$ that satisfies $\mathbf{X}''(x) = -\lambda\mathbf{X}(x)$ and that

- satisfies boundary conditions $\mathbf{X}(0) = \mathbf{X}(L) = 0$,
- is NOT identically zero.

Determine values of λ: trivial cases

Now find $\mathbf{X}(x)$ that satisfies $\mathbf{X}''(x) = -\lambda \mathbf{X}(x)$ and that

- satisfies boundary conditions $\mathbf{X}(0) = \mathbf{X}(L) = 0$,
- is NOT identically zero.

There are two trivial cases:

- If $\lambda = 0$, then $\mathbf{X}(x) = A\,x + B$ for constants A and B. By boundary conditions,

$$A\cdot 0 + B = A\,L + B = 0, \implies A = B = 0, \boxed{\text{NOT non-zero solution}}.$$

- If $\lambda < 0$, then $\mathbf{X}(x) = A\,e^{\sqrt{-\lambda}\,x} + B\,e^{-\sqrt{-\lambda}\,x}$ for constants A and B. By boundary conditions,

$$A + B = A\,e^{\sqrt{-\lambda}\,L} + B\,e^{-\sqrt{-\lambda}\,L} = 0, \implies A = B = 0,$$

$$\boxed{\text{NOT non-zero solution}}.$$

Determine values of λ: eigenvalue cases

Now find $\mathbf{X}(x)$ that satisfies $\mathbf{X}''(x) = -\lambda \mathbf{X}(x)$ <u>with $\lambda > 0$</u> and that

- satisfies boundary conditions $\mathbf{X}(0) = \mathbf{X}(L) = 0$,
- is NOT identically zero.

Determine values of λ: eigenvalue cases

Now find $\mathbf{X}(x)$ that satisfies $\mathbf{X}''(x) = -\lambda\,\mathbf{X}(x)$ with $\lambda > 0$ and that

- satisfies boundary conditions $\mathbf{X}(0) = \mathbf{X}(L) = 0$,
- is NOT identically zero.

$$\mathbf{X}(x) = A\cos\left(\sqrt{\lambda}\,x\right) + B\sin\left(\sqrt{\lambda}\,x\right)$$

for constants A and B. By boundary conditions,

$$\begin{aligned} A\cos\left(\sqrt{\lambda}\cdot 0\right) + B\sin\left(\sqrt{\lambda}\cdot 0\right) &= 0, \quad \Longrightarrow A = 0, \\ A\cos\left(\sqrt{\lambda}\,L\right) + B\sin\left(\sqrt{\lambda}\,L\right) &= 0, \quad \Longrightarrow B\sin\left(\sqrt{\lambda}\,L\right) = 0. \end{aligned}$$

Last equation possible only when $\sqrt{\lambda}\,L = n\,\pi$ for positive integers $n = 1, 2, 3, \cdots$,

Thus $\lambda = \left(\frac{n\,\pi}{L}\right)^2$, with $\mathbf{X}(x) = B\sin\left(\frac{n\,\pi\,x}{L}\right)$ for $n = 1, 2, 3, \cdots$,

Determine particular solutions

Now find $u(x,t) = \mathbf{X}(x)\,\mathbf{T}(t)$ for $\mathbf{X}(x) = B\,\mathbf{sin}\left(\sqrt{\lambda}\,x\right)$ with $\lambda = \left(\frac{n\,\pi}{L}\right)^2$ for $n = 1, 2, 3, \cdots$,

- $\mathbf{T}(t)$ satisfies

$$\mathbf{T}'(t) = -\beta\,\lambda\,\mathbf{T}(t), \quad \mathbf{T}(t) = C\,e^{-\beta\,\lambda\,t}.$$

- Particular solution $u_n(x,t) \stackrel{def}{=} e^{-\beta\left(\frac{n\,\pi}{L}\right)^2 t}\,\mathbf{sin}\left(\frac{n\,\pi\,x}{L}\right)$ for $n = 1, 2, 3, \cdots$,

Determine particular solutions

Now find $u(x,t) = \mathbf{X}(x)\ \mathbf{T}(t)$ for $\mathbf{X}(x) = B\,\mathbf{sin}\left(\sqrt{\lambda}\,x\right)$ with $\lambda = \left(\frac{n\pi}{L}\right)^2$ for $n = 1, 2, 3, \cdots$,

- $\mathbf{T}(t)$ satisfies

$$\mathbf{T}'(t) = -\beta\,\lambda\,\mathbf{T}(t), \quad \mathbf{T}(t) = C\,e^{-\beta\,\lambda\,t}.$$

- Particular solution $u_n(x,t) \stackrel{def}{=} e^{-\beta\left(\frac{n\pi}{L}\right)^2 t}\,\mathbf{sin}\left(\frac{n\pi x}{L}\right)$ for $n = 1, 2, 3, \cdots$,

$u_n(x,t)$ satisfies differential equation and boundary conditions

$$\begin{aligned} \frac{\partial u_n}{\partial t} &= \beta\,\frac{\partial^2 u_n}{\partial x^2}, \quad 0 < x < L, \quad t > 0. \quad (\ell) \\ u_n(0,t) &= 0 = u_n(L,t) \quad \forall\, t > 0. \end{aligned}$$

Determine particular solutions

Now find $u(x,t) = \mathbf{X}(x)\,\mathbf{T}(t)$ for $\mathbf{X}(x) = B\,\mathbf{sin}\left(\sqrt{\lambda}\,x\right)$ with $\lambda = \left(\frac{n\,\pi}{L}\right)^2$ for $n = 1, 2, 3, \cdots$,

- $\mathbf{T}(t)$ satisfies

$$\mathbf{T}'(t) = -\beta\,\lambda\,\mathbf{T}(t), \quad \mathbf{T}(t) = C\,e^{-\beta\,\lambda\,t}.$$

- Particular solution $u_n(x,t) \overset{def}{=} e^{-\beta\left(\frac{n\,\pi}{L}\right)^2 t}\,\mathbf{sin}\left(\frac{n\,\pi\,x}{L}\right)$ for $n = 1, 2, 3, \cdots$,

$u_n(x,t)$ satisfies differential equation and boundary conditions

$$\begin{aligned} \frac{\partial u_n}{\partial t} &= \beta\,\frac{\partial^2 u_n}{\partial x^2}, \quad 0 < x < L, \quad t > 0. \quad (\ell) \\ u_n(0,t) &= 0 = u_n(L,t) \quad \forall\, t > 0. \end{aligned}$$

Ditto any convergent series $\sum_{n=1}^{\infty} \mu_n\,e^{-\beta\left(\frac{n\,\pi}{L}\right)^2 t}\,\mathbf{sin}\left(\frac{n\,\pi\,x}{L}\right)$.

Solve Heat Equation

Let $u(x,t) = \sum_{n=1}^{\infty} \mu_n \, e^{-\beta \left(\frac{n\pi}{L}\right)^2 t} \, \mathbf{sin}\left(\frac{n\pi x}{L}\right)$ solve heat equation

$$\begin{aligned} \frac{\partial u}{\partial t} &= \beta \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L, \quad t > 0. \quad (\ell) \\ u(x,0) &= f(x) \quad \forall\, x \in [0, L]; \qquad u(0,t) = 0 = u(L,t) \quad \forall\, t > 0. \end{aligned}$$

Solve Heat Equation

Let $u(x,t) = \sum_{n=1}^{\infty} \mu_n \, e^{-\beta \left(\frac{n\pi}{L}\right)^2 t} \, \mathbf{sin}\left(\frac{n\pi x}{L}\right)$ solve heat equation

$$
\begin{aligned}
\frac{\partial u}{\partial t} &= \beta \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L, \quad t > 0. \quad (\ell) \\
u(x,0) &= f(x) \quad \forall\, x \in [0,L]; \qquad u(0,t) = 0 = u(L,t) \quad \forall\, t > 0.
\end{aligned}
$$

Thus

$$
f(x) = u(x,0) = \sum_{n=1}^{\infty} \mu_n \, \mathbf{sin}\left(\frac{n\pi x}{L}\right), \quad \longleftarrow \text{Fourier Sine series.}
$$

$$
\text{Therefore} \quad \mu_n = \frac{2}{L} \int_0^L f(x) \, \mathbf{sin}\left(\frac{n\pi x}{L}\right) dx.
$$

Solve heat equation, **Example I**

$$\begin{aligned} \frac{\partial u}{\partial t} &= 2\,\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \pi, \quad t > 0, \\ u(x,0) &= \frac{\pi}{2} - \left|\frac{\pi}{2} - x\right| \quad \forall\, x \in [0,\pi]; \quad u(0,t) = 0 = u(\pi,t) \quad \forall\, t > 0. \end{aligned}$$

Solve heat equation, **Example I**

$$\begin{aligned}\frac{\partial u}{\partial t} &= 2\,\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \pi, \quad t > 0,\\ u(x,0) &= \frac{\pi}{2} - \left|\frac{\pi}{2} - x\right| \quad \forall\, x \in [0,\pi]; \quad u(0,t) = 0 = u(\pi,t) \quad \forall\, t > 0.\end{aligned}$$

SOLUTION: $\beta = 2$, $L = \pi$, and solution takes form

$$u(x,t) = \sum_{n=1}^{\infty} \mu_n\, e^{-2\,n^2\,t}\, \mathbf{sin}\,(n\,x)\,, \quad \text{where}$$

$$\mu_n = \frac{2}{\pi}\int_0^{\pi} \left(\frac{\pi}{2} - \left|\frac{\pi}{2} - x\right|\right) \mathbf{sin}\,(n\,x)\,dx = \begin{cases} 0, & \text{if } n \text{ is even,}\\ \frac{4\,(-1)^{(n-1)/2}}{\pi\, n^2} & \text{if } n \text{ is odd.}\end{cases}$$

$$u(x,t) = \frac{4}{\pi}\left(e^{-2\,t}\,\mathbf{sin}\,(x) - \frac{e^{-18\,t}}{9}\,\mathbf{sin}\,(3\,x) + \frac{e^{-50\,t}}{25}\,\mathbf{sin}\,(5\,x) + \cdots\right)$$

Solve heat equation, **Example I**

$$\begin{aligned} \frac{\partial u}{\partial t} &= 2\,\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \pi, \quad t > 0, \\ u(x,0) &= \frac{\pi}{2} - \left|\frac{\pi}{2} - x\right| \quad \forall\, x \in [0,\pi]; \;\; u(0,t) = 0 = u(\pi,t) \;\; \forall\, t > 0. \end{aligned}$$

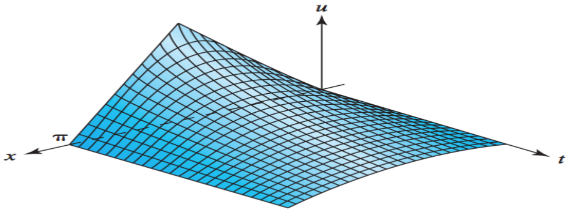
Solve heat equation, **Example I**

$$\frac{\partial u}{\partial t} = 2\,\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \pi, \quad t > 0,$$

$$u(x,0) = \frac{\pi}{2} - \left|\frac{\pi}{2} - x\right| \quad \forall\, x \in [0,\pi]; \quad u(0,t) = 0 = u(\pi,t) \quad \forall\, t > 0.$$

SOLUTION:

$$u(x,t) = \frac{4}{\pi}\left(e^{-2\,t}\,\mathbf{sin}\,(x) - \frac{e^{-18\,t}}{9}\,\mathbf{sin}\,(3\,x) + \frac{e^{-50\,t}}{25}\,\mathbf{sin}\,(5\,x) + \cdots\right)$$



Heat conduction, no heat flow at ends

$$\begin{aligned} \frac{\partial u}{\partial t} &= \beta \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L, \quad t > 0. \quad (\ell) \\ u(x,0) &= f(x) \quad \forall\, x \in [0,L]; \quad \frac{\partial u}{\partial x}(0,t) = 0 = \frac{\partial u}{\partial x}(L,t) \quad \forall\, t > 0. \end{aligned}$$

Heat conduction, no heat flow at ends

$$\begin{aligned} \frac{\partial u}{\partial t} &= \beta \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L, \quad t > 0. \quad (\ell) \\ u(x,0) &= f(x) \quad \forall\, x \in [0,L]; \quad \frac{\partial u}{\partial x}(0,t) = 0 = \frac{\partial u}{\partial x}(L,t) \quad \forall\, t > 0. \end{aligned}$$

SEPARATION OF VARIABLES: First try to find a solution in form

$$u(x,t) = \mathbf{X}(x)\,\mathbf{T}(t) \quad \text{that}$$

- satisfies boundary conditions $\mathbf{X}'(0) = \mathbf{X}'(L) = 0$,
- is NOT identically zero.

Heat conduction, no heat flow at ends

$$\begin{aligned} \frac{\partial u}{\partial t} &= \beta \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L, \quad t > 0. \quad (\ell) \\ u(x,0) &= f(x) \quad \forall\, x \in [0,L]; \quad \frac{\partial u}{\partial x}(0,t) = 0 = \frac{\partial u}{\partial x}(L,t) \quad \forall\, t > 0. \end{aligned}$$

SEPARATION OF VARIABLES: First try to find a solution in form

$$u(x,t) = \mathbf{X}(x)\,\mathbf{T}(t) \quad \text{that}$$

- satisfies boundary conditions $\mathbf{X}'(0) = \mathbf{X}'(L) = 0$,
- is NOT identically zero.

From $\dfrac{\partial u}{\partial t} = \mathbf{X}(x)\,\mathbf{T}'(t)$, $\dfrac{\partial^2 u}{\partial x^2} = \mathbf{X}''(x)\,\mathbf{T}(t)$

Heat conduction, no heat flow at ends

$$\begin{aligned}\frac{\partial u}{\partial t} &= \beta\,\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L, \quad t > 0. \quad (\ell)\\ u(x,0) &= f(x) \quad \forall\, x \in [0,L]; \quad \frac{\partial u}{\partial x}(0,t) = 0 = \frac{\partial u}{\partial x}(L,t) \quad \forall\, t > 0.\end{aligned}$$

SEPARATION OF VARIABLES: First try to find a solution in form

$$u(x,t) = \mathbf{X}(x)\ \mathbf{T}(t) \quad \text{that}$$

- satisfies boundary conditions $\mathbf{X}'(0) = \mathbf{X}'(L) = 0$,
- is NOT identically zero.

From $\dfrac{\partial u}{\partial t} = \mathbf{X}(x)\ \mathbf{T}'(t)$, $\dfrac{\partial^2 u}{\partial x^2} = \mathbf{X}''(x)\ \mathbf{T}(t)$

$$\implies \quad \mathbf{X}(x)\ \mathbf{T}'(t) = \beta\,\mathbf{X}''(x)\ \mathbf{T}(t)$$

Heat conduction, no heat flow at ends

$$\begin{aligned} \frac{\partial u}{\partial t} &= \beta \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L, \quad t > 0. \quad (\ell) \\ u(x,0) &= f(x) \quad \forall\, x \in [0,L]; \quad \frac{\partial u}{\partial x}(0,t) = 0 = \frac{\partial u}{\partial x}(L,t) \quad \forall\, t > 0. \end{aligned}$$

Separation of Variables: First try to find a solution in form

$$u(x,t) = \mathbf{X}(x)\ \mathbf{T}(t) \quad \text{that}$$

- satisfies boundary conditions $\mathbf{X}'(0) = \mathbf{X}'(L) = 0$,
- is NOT identically zero.

$$\text{From} \quad \frac{\partial u}{\partial t} = \mathbf{X}(x)\ \mathbf{T}'(t), \quad \frac{\partial^2 u}{\partial x^2} = \mathbf{X}''(x)\ \mathbf{T}(t)$$

$$\Longrightarrow \quad \mathbf{X}(x)\ \mathbf{T}'(t) = \beta\, \mathbf{X}''(x)\ \mathbf{T}(t) \quad \Longrightarrow \quad \frac{\mathbf{X}''(x)}{\mathbf{X}(x)} = \frac{\mathbf{T}'(t)}{\beta\, \mathbf{T}(t)} \overset{def}{=} -\lambda.$$

λ: neither function of x nor t, therefore must be certain constant

Determine values of λ: trivial case

Now find $\mathbf{X}(x)$ that satisfies $\mathbf{X}''(x) = -\lambda \mathbf{X}(x)$ and that

- satisfies boundary conditions $\mathbf{X}'(0) = \mathbf{X}'(L) = 0$,
- is NOT identically zero.

Determine values of λ: trivial case

Now find $\mathbf{X}(x)$ that satisfies $\mathbf{X}''(x) = -\lambda\,\mathbf{X}(x)$ and that

- satisfies boundary conditions $\mathbf{X}'(0) = \mathbf{X}'(L) = 0$,
- is NOT identically zero.

There is one trivial case:

- If $\lambda < 0$, then $\mathbf{X}(x) = A\,e^{\sqrt{-\lambda}\,x} + B\,e^{-\sqrt{-\lambda}\,x}$ for constants A and B. By boundary conditions,

$$\sqrt{-\lambda}\,(A - B) = \sqrt{-\lambda}\left(A\,e^{\sqrt{-\lambda}\,L} - B\,e^{-\sqrt{-\lambda}\,L}\right) = 0,$$

$$\Longrightarrow A = B = 0, \quad \boxed{\text{NOT non-zero solution}}.$$

Determine values of λ: eigenvalue cases (I)

- If $\lambda = 0$, then $\mathbf{X}(x) = A + B\,x$ for constants A and B. By boundary conditions, $B = 0$.
  Solution $\mathbf{X}(x) = A$ for arbitrary constant A.

Determine values of λ: eigenvalue cases (II)

- If $\lambda > 0$, then

$$\mathbf{X}(x) = A\,\mathbf{cos}\left(\sqrt{\lambda}\,x\right) + B\,\mathbf{sin}\left(\sqrt{\lambda}\,x\right) \quad \text{for constants } A \text{ and } B.$$

By boundary conditions $\mathbf{X}'(0) = \mathbf{X}'(L) = 0$,

$$\begin{aligned}
\sqrt{\lambda}\left(-A\,\mathbf{sin}\left(\sqrt{\lambda}\cdot 0\right) + B\,\mathbf{cos}\left(\sqrt{\lambda}\cdot 0\right)\right) &= 0, \quad \Longrightarrow B = 0, \\
\sqrt{\lambda}\left(-A\,\mathbf{sin}\left(\sqrt{\lambda}\,L\right) + B\,\mathbf{cos}\left(\sqrt{\lambda}\,L\right)\right) &= 0, \quad \Longrightarrow A\,\mathbf{sin}\left(\sqrt{\lambda}\,L\right) = 0.
\end{aligned}$$

Last equation possible only when $\sqrt{\lambda}\,L = n\,\pi$
for positive integers $n = 1, 2, 3, \cdots$. Together with the case $\lambda = 0$,

$$\lambda = \left(\frac{n\,\pi}{L}\right)^2, \quad \text{with } \mathbf{X}(x) = B\,\mathbf{cos}\left(\frac{n\,\pi\,x}{L}\right) \quad \text{for} \quad n = 0, 1, 2, 3, \cdots,$$

Determine particular solutions

Now find $u(x,t) = \mathbf{X}(x)\,\mathbf{T}(t)$ for $\mathbf{X}(x) = B\,\mathbf{cos}\left(\sqrt{\lambda}\,x\right)$ with $\lambda = \left(\frac{n\,\pi}{L}\right)^2$ for $n = 0, 1, 2, 3, \cdots$,

- $\mathbf{T}(t)$ satisfies

$$\mathbf{T}'(t) = -\beta\,\lambda\,\mathbf{T}(t)\,, \quad \mathbf{T}(t) = C\,e^{-\beta\,\lambda\,t}.$$

- Particular solution $u_0(x,t) \stackrel{def}{=} 1$ and $u_n(x,t) \stackrel{def}{=} e^{-\beta\left(\frac{n\,\pi}{L}\right)^2 t}\,\mathbf{cos}\left(\frac{n\,\pi\,x}{L}\right)$ for $n = 1, 2, 3, \cdots$,

Determine particular solutions

Now find $u(x,t) = \mathbf{X}(x)\,\mathbf{T}(t)$ for $\mathbf{X}(x) = B\,\mathbf{cos}\left(\sqrt{\lambda}\,x\right)$ with $\lambda = \left(\frac{n\,\pi}{L}\right)^2$ for $n = 0, 1, 2, 3, \cdots$,

- $\mathbf{T}(t)$ satisfies

$$\mathbf{T}'(t) = -\beta\,\lambda\,\mathbf{T}(t), \quad \mathbf{T}(t) = C\,e^{-\beta\,\lambda\,t}.$$

- Particular solution $u_0(x,t) \stackrel{def}{=} 1$ and $u_n(x,t) \stackrel{def}{=} e^{-\beta\left(\frac{n\,\pi}{L}\right)^2 t}\,\mathbf{cos}\left(\frac{n\,\pi\,x}{L}\right)$ for $n = 1, 2, 3, \cdots$,

$u_n(x,t)$ satisfies differential equation and boundary conditions

$$\begin{aligned} \frac{\partial u_n}{\partial t} &= \beta\,\frac{\partial^2 u_n}{\partial x^2}, \quad 0 < x < L, \quad t > 0. \quad (\ell) \\ u_n(0,t) &= 0 = u_n(L,t) \quad \forall\, t > 0. \end{aligned}$$

Determine particular solutions

Now find $u(x,t) = \mathbf{X}(x)\,\mathbf{T}(t)$ for $\mathbf{X}(x) = B\,\mathbf{cos}\left(\sqrt{\lambda}\,x\right)$ with $\lambda = \left(\frac{n\,\pi}{L}\right)^2$ for $n = 0, 1, 2, 3, \cdots$,

- $\mathbf{T}(t)$ satisfies

$$\mathbf{T}'(t) = -\beta\,\lambda\,\mathbf{T}(t), \quad \mathbf{T}(t) = C\,e^{-\beta\,\lambda\,t}.$$

- Particular solution $u_0(x,t) \stackrel{def}{=} 1$ and $u_n(x,t) \stackrel{def}{=} e^{-\beta\left(\frac{n\,\pi}{L}\right)^2 t}\,\mathbf{cos}\left(\frac{n\,\pi\,x}{L}\right)$ for $n = 1, 2, 3, \cdots$,

$u_n(x,t)$ satisfies differential equation and boundary conditions

$$\begin{aligned} \frac{\partial u_n}{\partial t} &= \beta\,\frac{\partial^2 u_n}{\partial x^2}, \quad 0 < x < L, \quad t > 0. \quad (\ell) \\ u_n(0,t) &= 0 = u_n(L,t) \quad \forall\, t > 0. \end{aligned}$$

Ditto any convergent series $\frac{\mu_0}{2} + \sum_{n=1}^{\infty} \mu_n\,e^{-\beta\left(\frac{n\,\pi}{L}\right)^2 t}\,\mathbf{cos}\left(\frac{n\,\pi\,x}{L}\right)$.

Solve Heat Equation

Let $u(x,t) = \frac{\mu_0}{2} + \sum_{n=1}^{\infty} \mu_n \, e^{-\beta \left(\frac{n\pi}{L}\right)^2 t} \, \mathbf{cos}\left(\frac{n\pi x}{L}\right)$ solve heat equation

$$\begin{aligned} \frac{\partial u}{\partial t} &= \beta \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L, \quad t > 0. \quad (\ell) \\ u(x,0) &= f(x) \quad \forall\, x \in [0,L]; \quad \frac{\partial u}{\partial x}(0,t) = 0 = \frac{\partial u}{\partial x}(L,t) \quad \forall\, t > 0. \end{aligned}$$

Solve Heat Equation

Let $u(x,t) = \frac{\mu_0}{2} + \sum_{n=1}^{\infty} \mu_n \, e^{-\beta \left(\frac{n\pi}{L}\right)^2 t} \, \mathbf{cos}\left(\frac{n\pi x}{L}\right)$ solve heat equation

$$\begin{aligned} \frac{\partial u}{\partial t} &= \beta \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L, \quad t > 0. \quad (\ell) \\ u(x,0) &= f(x) \quad \forall\, x \in [0, L]; \quad \frac{\partial u}{\partial x}(0,t) = 0 = \frac{\partial u}{\partial x}(L,t) \quad \forall\, t > 0. \end{aligned}$$

Thus

$$f(x) = u(x,0) = \frac{\mu_0}{2} + \sum_{n=1}^{\infty} \mu_n \, \mathbf{cos}\left(\frac{n\pi x}{L}\right), \quad \longleftarrow \text{Fourier Cosine series.}$$

Therefore $\mu_n = \frac{2}{L} \int_0^L f(x) \, \mathbf{cos}\left(\frac{n\pi x}{L}\right) dx, \quad n = 0, 1, 2, \cdots$

Solve heat equation, **Example II**

$$\begin{aligned} \frac{\partial u}{\partial t} &= \beta \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L, \quad t > 0. \quad (\ell) \\ u(x,0) &= \frac{\pi}{2} - \left| \frac{\pi}{2} - x \right| \ \ \forall\, x \in [0, L]; \ \ \frac{\partial u}{\partial x}(0,t) = 0 = \frac{\partial u}{\partial t}(L,t) \ \ \forall\, t > 0. \end{aligned}$$

Solve heat equation, **Example II**

$$\begin{aligned}\frac{\partial u}{\partial t} &= \beta\,\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L, \quad t > 0. \quad (\ell)\\ u(x,0) &= \frac{\pi}{2} - \left|\frac{\pi}{2} - x\right| \quad \forall\, x \in [0, L]; \quad \frac{\partial u}{\partial x}(0,t) = 0 = \frac{\partial u}{\partial t}(L,t) \quad \forall\, t > 0.\end{aligned}$$

SOLUTION: $\beta = 2$, $L = \pi$, and solution takes form

$$u(x,t) = \frac{\mu_0}{2} + \sum_{n=1}^{\infty} \mu_n\, e^{-2\,n^2\,t}\,\mathbf{cos}\,(n\,x)\,, \quad \text{where}$$

$$\mu_n = \frac{2}{\pi}\int_0^{\pi} \left(\frac{\pi}{2} - \left|\frac{\pi}{2} - x\right|\right)\mathbf{cos}\,(n\,x)\,dx = \begin{cases} \frac{\pi}{2}, & \text{if } n = 0,\\ 0, & \text{if } n \text{ is odd},\\ \frac{(-1)^k - 1}{2\,k^2}, & \text{if } n = 2\,k \text{ even}.\end{cases}$$

$$u(x,t) = \frac{\pi}{4} - \frac{2}{\pi}\left(e^{-8\,t}\,\mathbf{cos}\,(2\,x) + \frac{e^{-72\,t}}{9}\,\mathbf{cos}\,(6\,x) + \cdots\right)$$

Constant boundary temps, **Example III**

$$\begin{aligned} \frac{\partial u}{\partial t} &= \beta \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L, \quad t > 0. \\ u(x,0) &= f(x) \ \forall\, x \in [0,L]; \quad u(0,t) = \underbrace{U_0,\ \ u(L,t) = U_1}_{\text{boundary temps}} \ \forall\, t > 0. \end{aligned}$$

Constant boundary temps, **Example III**

$$\begin{aligned}\frac{\partial u}{\partial t} &= \beta \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L, \quad t > 0.\\ u(x,0) &= f(x)\ \forall\, x \in [0,L]; \quad u(0,t) = \underbrace{U_0}, \ u(L,t) = \underbrace{U_1} \ \forall\, t > 0.\end{aligned}$$

boundary temps

SEPARATION OF STEADY-STATE AND TRANSIENT: First let

$$u(x,t) = v(x) + w(x,t), \text{ with } v(x) = U_0 + \tfrac{x}{L}(U_1 - U_0).$$

$$\text{Then } \frac{\partial v}{\partial t} = \beta \frac{\partial^2 v}{\partial x^2}; \quad v(0,t) = U_0, \ v(L,t) = U_1 \quad \text{and}$$

$$\begin{aligned}\frac{\partial w}{\partial t} &= \beta \frac{\partial^2 w}{\partial x^2}, \quad 0 < x < L, \quad t > 0. \quad (\ell)\\ w(x,0) &= f(x) - v(x)\ \forall\, x \in [0,L]; \quad w(0,t) = 0 = w(L,t) \ \forall\, t > 0\end{aligned}$$

Constant boundary temps with source, **Example IV**

$$\begin{aligned} \frac{\partial u}{\partial t} &= \beta \frac{\partial^2 u}{\partial x^2} + \overbrace{P(x)}^{\text{time-independent source}}, \quad 0 < x < L, \quad t > 0. \quad (\ell) \\ u(x,0) &= f(x) \ \forall\, x \in [0,L]; \quad u(0,t) = \underbrace{U_0,\ u(L,t) = U_1}_{\text{boundary temps}} \ \forall\, t > 0. \end{aligned}$$

Constant boundary temps with source, **Example IV**

$$\begin{aligned}\frac{\partial u}{\partial t} &= \beta \frac{\partial^2 u}{\partial x^2} + \overbrace{P(x)}^{\text{time-independent source}}, \quad 0 < x < L, \quad t > 0. \quad (\ell) \\ u(x,0) &= f(x)\ \forall\, x \in [0,L]; \quad u(0,t) = \underbrace{U_0}_{\text{boundary temps}},\ u(L,t) = \underbrace{U_1}\ \ \forall\, t > 0.\end{aligned}$$

Set $u(x,t) = v(x) + w(x,t)$, with

$$v(x) = U_0 + \frac{x}{L}(U_1 - U_0) + \int_0^x \frac{s}{\beta}\left(1 - \frac{x}{L}\right) P(s)\, ds + \int_x^L \frac{x}{\beta}\left(1 - \frac{s}{L}\right) P(s)\, ds.$$

Then $\dfrac{\partial v}{\partial t} = \beta \dfrac{\partial^2 v}{\partial x^2} + P(x);\quad v(0,t) = U_0,\ \ v(L,t) = U_1$ and

$$\begin{aligned}\frac{\partial w}{\partial t} &= \beta \frac{\partial^2 w}{\partial x^2}, \quad 0 < x < L, \quad t > 0. \\ w(x,0) &= f(x) - v(x)\ \forall\, x \in [0,L]; \quad w(0,t) = 0 = w(L,t)\ \ \forall\, t > 0\end{aligned}$$