Quiz 9

NOVEMBER 14, 2019

Name (Last, First)

Key, Answer

1. Let A be the following symmetric matrix:

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \\ 2 & 2 & 3 \end{bmatrix}.$$

It is known that the characteristic polynomial of A is

$$\chi_\lambda(A) = \det(A - \lambda I) = (1+\lambda)^2(5-\lambda).$$

a) (4pts) Find an **orthogonal** basis of $\mathbb{R}^3$ consisting of eigenvectors of A.

There are two distinct eigenvalues : $\lambda = -1$ and $\lambda = 5$.

$$E_{\lambda=-1} = \operatorname{Nul}(A-(-1)I) = \operatorname{Nul}\begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 4 \end{bmatrix} = \operatorname{Nul}\begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \operatorname{Span}\left\{\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\right\}$$

$$E_{\lambda=5} = \operatorname{Nul}(A-5I) = \operatorname{Nul}\begin{bmatrix} -5 & 1 & 2 \\ 1 & -5 & 2 \\ 2 & 2 & -2 \end{bmatrix} = \operatorname{Nul}\begin{bmatrix} 1 & 1 & -1 \\ 1 & -5 & 2 \\ -5 & 1 & 2 \end{bmatrix} = \operatorname{Nul}\begin{bmatrix} 1 & 1 & -1 \\ 0 & -6 & 3 \\ * & * & * \end{bmatrix} = \operatorname{Span}\left\{\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}\right\}$$

$\leftarrow$ will be dependent

As $-1 \neq 5$, $E_{\lambda=-1} \perp E_{\lambda=5}$. We only need to make $E_{\lambda=-1}$'s basis orthogonal.

Set $v_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$ and apply Gram-Schmidt.

v_2 will be modified into $\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} - \dfrac{v_1 \cdot v_2}{v_1 \cdot v_1} v_1 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} - v_1 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$.

Answer: $\mathcal{B} = \left\{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}\right\}$.

b) (3pts) Orthogonally diagonalize A. (In other words, find an orthogonal matrix P and a diagonal matrix D such that $A = PDP^{-1}$. Here, you should compute what P^{-1} is.) [Please recall that "orthogonal matrix" does not just mean that it has orthogonal columns but the definition is a bit stronger. This fact will make the computation for P^{-1} very easy.]

After changing $\mathcal{B}$ into an orthonormal basis, you can combine the vectors to get P. Normalized $\mathcal{B}$ is $\left\{\frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{3}}\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}\right\}$.

$$A = PDP^{-1} = \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{3} & 1/\sqrt{6} \\ 1/\sqrt{2} & -1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & 2/\sqrt{6} \end{bmatrix} \begin{bmatrix} -1 & & \\ & -1 & \\ & & 5 \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{2} & 0 \\ -1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{6} & 2/\sqrt{6} \end{bmatrix}$$

(Note that P^{-1} is the same as P^T because the columns of P are orthonormal. So, we used it here.)

(See the reverse side.)

(part b continued)

$v_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}.$

c) (3pts) Suppose that your answer for part a be $\{v_1, v_2, v_3\}$ where v_1 and v_2 are eigenvectors corresponding to $\lambda = -1$ and v_3 is an eigenvector corresponding to $\lambda = 5$. Compute the following sum of matrices (note that the numerators are 3×3 matrices and the denominators are just real numbers):

$$(-1)\frac{v_1 v_1^T}{v_1^T v_1} + (-1)\frac{v_2 v_2^T}{v_2^T v_2} + 5\frac{v_3 v_3^T}{v_3^T v_3}.$$

$$\frac{-\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}{2} + \frac{-\begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}}{3} + \frac{5 \cdot \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 4 \end{bmatrix}}{6}$$

$$= \frac{1}{6}\left(\begin{bmatrix} -3 & 3 & 0 \\ 3 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} -2 & -2 & 2 \\ -2 & -2 & 2 \\ 2 & 2 & -2 \end{bmatrix} + \begin{bmatrix} 5 & 5 & 10 \\ 5 & 5 & 10 \\ 10 & 10 & 20 \end{bmatrix} \right)$$

$$= \frac{1}{6}\begin{bmatrix} 0 & 6 & 12 \\ 6 & 0 & 12 \\ 12 & 12 & 18 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \\ 2 & 2 & 3 \end{bmatrix} = \text{the original } A.$$

(This result is called the Spectral Decomposition.)