Quiz 8

OCTOBER 31, 2019

Name (Last, First)

Answer Key

1. (6pts) Let A be the following matrix:

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 3 & 2 & 4 \\ 1 & -1 & -2 & 0 \end{bmatrix}.$$

a) Find a basis of Col$A$. (Fact: The pivot columns form a basis of Col$A$.)

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 3 & 2 & 4 \\ 1 & -1 & -2 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 4 & 4 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

pivot columns

pivot columns form a basis of Col(A)

$\hookrightarrow$ Span $\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} \right\}$

b) Using Gram-Schmidt process, find an orthogonal basis of Col$A$ and find $\text{proj}_{\text{Col}A}\mathbf{y}$ (the orthogonal projection of $\mathbf{y}$ onto Col$A$) where $\mathbf{y} = \begin{bmatrix} -1 \\ 0 \\ 4 \end{bmatrix}$.

$x_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, $x_2 = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$

Gram-Schmidt:

orthogonal basis of Col A = Span $\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} \right\}$

$$V_1 = X_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$V_2 = X_2 - \text{proj}_{V_1} X_2 = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} - \frac{\vec{x_2} \cdot \vec{v_1}}{\vec{v_1} \cdot \vec{v_1}} \vec{v_1} = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} - \frac{0 \cdot 1 + 1 \cdot 3 + 1 \cdot -1}{0 \cdot 0 + 1 \cdot 1 + 1 \cdot 1} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$$

$\text{proj}_{\text{col}A}\vec{y}$:

① $\begin{bmatrix} 0 & 1 & -1 \\ 1 & 3 & 0 \\ 1 & -1 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 0 \\ 0 & 4 & -4 \\ 0 & 1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

$\hookrightarrow$ consistent, $\vec{y}$ is spanned by the column space.

② $\text{proj}_{\text{col}A}\vec{y} = P(P^TP)^{-1}P^T\vec{y}$ where P is $\begin{bmatrix} 0 & 1 \\ 1 & 3 \\ 1 & -1 \end{bmatrix}$

$$= \begin{bmatrix} 0 & 1 \\ 1 & 3 \\ 1 & -1 \end{bmatrix} \left(\begin{bmatrix} 0 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 3 \\ 1 & -1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 3 \\ 1 & -1 \end{bmatrix} \left(\begin{bmatrix} 2 & 2 \\ 2 & 11 \end{bmatrix} \right)^{-1} \begin{bmatrix} 4 \\ -5 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ 1 & 3 \\ 1 & -1 \end{bmatrix} \left(\frac{1}{18} \begin{bmatrix} 11 & -2 \\ -2 & 2 \end{bmatrix} \right) \begin{bmatrix} 4 \\ -5 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 3 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 54/18 \\ -18/18 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 3 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 4 \end{bmatrix}$$

the projection of the vector stays the same as the original vector, so the vector $\vec{y}$ must be in the span of the column space.

2. (4pts) Find the normal equation for $B\mathbf{x} = b$ and find the **least-squares solution** of the equation where

$$B = \begin{bmatrix} 0 & 1 \\ 1 & 3 \\ 1 & -1 \end{bmatrix} \text{ and } b = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}.$$

Normal Equation:

$(B^TB)x = B^T\vec{b}$

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 3 \\ 1 & -1 \end{bmatrix} x = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

$$x = (B^TB)^{-1}B^T\vec{b}$$

$$= \left(\begin{bmatrix} 0 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 3 \\ 1 & -1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

$$= \left(\begin{bmatrix} 2 & 2 \\ 2 & 11 \end{bmatrix} \right)^{-1} \begin{bmatrix} 4 \\ 5 \end{bmatrix} = \frac{1}{18} \begin{bmatrix} 11 & -2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \end{bmatrix} = \frac{1}{18} \begin{bmatrix} 34 \\ 2 \end{bmatrix} = \begin{bmatrix} 34/18 \\ 1/9 \end{bmatrix} = \begin{bmatrix} 17/9 \\ 1/9 \end{bmatrix}$$

b) ③ $\text{proj}_{\text{col}A}\vec{y} = \frac{y\cdot\vec{v}_1}{\vec{v}_1\cdot\vec{v}_1}\vec{v}_1 + \frac{y\cdot\vec{v}_2}{\vec{v}_2\cdot\vec{v}_2}\vec{v}_2 = \frac{4}{2}\cdot\begin{bmatrix}0\\1\\1\end{bmatrix} + \frac{-9}{9}\begin{bmatrix}1\\2\\-2\end{bmatrix} = 2\begin{bmatrix}0\\1\\1\end{bmatrix} - 1\begin{bmatrix}1\\2\\-2\end{bmatrix} = \begin{bmatrix}-1\\0\\4\end{bmatrix}$

where $\vec{v}_1 = \begin{bmatrix}0\\1\\1\end{bmatrix}$ $\vec{y} = \begin{bmatrix}-1\\0\\4\end{bmatrix}$

$\vec{v}_2 = \begin{bmatrix}1\\2\\-2\end{bmatrix}$