Quiz 6

OCTOBER 17, 2019

Name (Last, First)

Answer Key

1. (7pts) Find all eigenvalues of the following matrix and, for each distinct eigenvalue, find one corresponding (non-zero) eigenvector.[1]

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & -1 & -1 \end{bmatrix}.$$

$$A - \lambda I = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & -1 & -1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} = \begin{bmatrix} 1-\lambda & 1 & 1 \\ 0 & 2-\lambda & 2 \\ 0 & -1 & -1-\lambda \end{bmatrix}$$

don't do row reduction when finding $\chi_A(\lambda)$

$\det(A - \lambda I) = 0$

$$\det\left(\begin{bmatrix} \cancel{1-\lambda} & \cancel{1} & \cancel{1} \\ \cancel{0} & 2-\lambda & 2 \\ \cancel{0} & -1 & -1-\lambda \end{bmatrix}\right) \rightarrow \text{cofactor expansion} = (1-\lambda)\left[(2-\lambda)(-1-\lambda)+2\right] = 0$$

$\chi_A(\lambda) = (1-\lambda)(-2-2\lambda+\lambda+\lambda^2+2) = (1-\lambda)(\lambda^2-\lambda) = 0$

$\lambda(\lambda-1)(1-\lambda) = 0$

characteristic equation

$\lambda = 0, 1$

for $\lambda = 0$,

$$A_0 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & -1 & -1 \end{bmatrix}$$

row reduce* to find nullspace

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 = 0$
$x_2 = -x_3$
$x_3 = x_3$

$$\text{Nul}(A_0) = \text{Span}\left\{\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}\right\}$$

$$\lambda = 0,\ \vec{v} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

for $\lambda = 1$,

$$A_1 = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & -1 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 = x_1$
$x_2 = 0$
$x_3 = 0$

$$\text{Nul}(A_1) = \text{Span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right\}$$

$$\lambda = 1,\ \vec{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

*be careful when doing row reduction

[1] Hint. The first column has many zeros.

2. (8pts) Diagonalize the following matrix. If impossible, explain why it is not diagonalizable.[2]

$$B = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ 0 & -1 & -1 \end{bmatrix}$$

→ even though A + B are very similar, A is not diagonalizable, but B is diagonalizable.

$$B - \lambda I = \begin{bmatrix} 1-\lambda & 1 & 2 \\ 0 & 2-\lambda & 2 \\ 0 & -1 & -1-\lambda \end{bmatrix}$$

$$\det(B - \lambda I) = (1-\lambda)[(2-\lambda)(-1-\lambda)+2] = 0$$

$\lambda = 0, 1$

↳ algebraic multiplicity of 2

for $\lambda = 0$,

$$B_0 = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ 0 & -1 & -1 \end{bmatrix}$$

↳ row reduce to find nullspace

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{aligned} x_1 &= -x_3 \\ x_2 &= -x_3 \\ x_3 &= x_3 \end{aligned}$$

$$\mathrm{Nul}(A) = \mathrm{Span}\left\{\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}\right\}$$

for $\lambda = 1$,

$$B_1 = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & -1 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{aligned} x_1 &= x_1 \\ x_2 &= -2x_3 \\ x_3 &= x_3 \end{aligned}$$

↓

1 eigenvalue can generate more than 1 eigenvectors

↳ compute $\dim E_\lambda$ by using rank theorem.

$$\mathrm{Nul}(A) = \mathrm{Span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}\right\}$$

$$P = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & -2 \\ 1 & 0 & 1 \end{bmatrix}$$

$\det(P) = -1(-1+2) = -1 \neq 0$

↳ P is invertible

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$B = PDP^{-1} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & -2 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & -2 \\ 1 & 0 & 1 \end{bmatrix}^{-1}$$

[2] Note that B looks similar to A from #1 but it is not.