Quiz 4

SEPTEMBER 19, 2019

Name (Last, First)

Answer Key

1. (7pts) Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ and $S : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformations defined by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -x+y \\ x-y \\ y \end{bmatrix} \quad \text{and} \quad S\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x+2y+z \\ 2x+2y+z \end{bmatrix}.$$

Please answer to the following 4 questions: Is T one-to-one? Is S one-to-one? Is T onto? Is S onto?

One-to-One (injective)

definition: If $x \neq y$, then $f(x) \neq f(y)$. Alternatively, for all y in the codomain, there is at most one x in the domain where $f(x)=y$.

condition: For $x \mapsto Ax$ to be one-to-one, Nul(A) must be zero. Therefore, every column in A must be linearly independent.

check: Check this condition by reducing the rows of A to see if there is a pivot in every column.

Onto (Surjective)

definition: A mapping $T: \mathbb{R}^n \to \mathbb{R}^m$ is onto $\mathbb{R}^m$ if each $\vec{b}$ in $\mathbb{R}^m$ is the image of at least one x in $\mathbb{R}^n$. For each y in the codomain, there is at least one x in the domain where $f(x)=y$.

condition: For $x \mapsto Ax$ to be onto, when you go from $\mathbb{R}^n \to \mathbb{R}^m$, the Col(A) = Span($\mathbb{R}^m$).

check: Check this condition by reducing the rows of A to see if there is a pivot in every row.

T:

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -x+y \\ x-y \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}x + \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}y$$

$$A_T = \begin{bmatrix} -1 & 1 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

pivots in every column, but not every row

$\therefore$ T is one-to-one but not onto.

S:

$$S\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+2y+z \\ 2x+2y+z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}x + \begin{bmatrix} 2 \\ 2 \end{bmatrix}y + \begin{bmatrix} 1 \\ 1 \end{bmatrix}z$$

$$A_S = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 \\ 0 & -2 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1/2 \end{bmatrix}$$

pivots in every row, but not in every column.

$\therefore$ T is onto, but not one-to-one.

2. (3pts) Compute the following product:

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} (-1\cdot1+2\cdot1+1\cdot0) & (1\cdot1+2\cdot-1+1\cdot1) \\ (-1\cdot2+2\cdot1+1\cdot0) & (2\cdot1+2\cdot-1+1\cdot1) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{bmatrix} \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \\ b_5 & b_6 \end{bmatrix} = \begin{bmatrix} a_1b_1+a_2b_3+a_3b_5 & a_1b_2+a_2b_4+a_3b_6 \\ a_4b_1+a_5b_3+a_6b_5 & a_4b_2+a_5b_4+a_6b_6 \end{bmatrix}$$

$$\underbrace{m_1 \times n_1 \cdot m_2}_{n_1 = m_2} \times n_2 = m_1 \times n_2 \text{ Sol. matrix}$$