Name (Last, First)

Answer Key

1. (7pts) Solve the initial value problem

$$y'' - 6y' + 9y = t^{-1}e^{3t}, \quad y(1) = 0, \quad y'(1) = 0.$$

auxillary equation

$r^2 - 6r + 9 = 0$

$(r-3)^2 = 0$

$r = 3$, multiplicity of 2

$y_h = e^{3t}, te^{3t}$ $\quad y_1 = e^{3t}, y_2 = te^{3t}$

$W = \det\left(\begin{bmatrix} e^{3t} & te^{3t} \\ 3e^{3t} & 3te^{3t} + e^{3t} \end{bmatrix}\right)$

$= 3te^{6t} + e^{6t} - 3te^{6t}$

$= e^{6t}$

$y_p = v_1 y_1 + v_2 y_2$

$v_1 = \int \frac{-y_2 f(t)}{W}\,dt$

$v_2 = \int \frac{y_1 f(t)}{W}\,dt$

$v_1 = \int \frac{-te^{3t} \cdot t^{-1}e^{3t}}{e^{6t}}\,dt = \int -1\,dt = -t$

$v_2 = \int \frac{e^{3t} \cdot t^{-1}e^{3t}}{e^{6t}}\,dt = \int \frac{1}{t}\,dt = \ln|t|$

$y_p = -te^{3t} + \ln|t| \cdot te^{3t}$

$y = -te^{3t} + \ln|t| \cdot te^{3t} + c_1 e^{3t} + c_2 te^{3t}$

$y' = -e^{3t} - 3te^{3t} + e^{3t} + \ln|t|(e^{3t} + 3te^{3t}) + 3c_1 e^{3t} + c_2 e^{3t} + 3c_2 te^{3t}$

$y(1) = -e^3 + c_1 e^3 + c_2 e^3 = 0$

$y'(1) = -e^3 - 3e^3 + e^3 + 3c_1 e^3 + 4c_2 e^3 = 0$

$\begin{cases} c_1 e^3 + c_2 e^3 = e^3 \\ 3c_1 e^3 + 4c_2 e^3 = 3e^3 \end{cases}$ $\quad c_1 = 1, \quad c_2 = 0$

$\boxed{y(t) = -te^{3t} + \ln|t|\, te^{3t} + e^{3t}}$

2. (3pts) Convert the differential equation in problem 1 into a normal form.

$x_1 = y(t)$

$x_2 = y'(t)$

$y(1) = 0$

$y'(1) = 0$

$\vec{x}(1) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\vec{x}'(t) = \begin{bmatrix} 0 & 1 \\ -c/a & -b/a \end{bmatrix} \vec{x}(t) + \begin{bmatrix} 0 \\ \frac{f(t)}{a} \end{bmatrix}$

$\vec{x}'(t) = \begin{bmatrix} 0 & 1 \\ -9 & 6 \end{bmatrix} \vec{x}(t) + \begin{bmatrix} 0 \\ t^{-1}e^{3t} \end{bmatrix}$

Thank YOU for a great semester.
Best, D.