Midterm 2 Practice

1) a. F, many vectors in V can be projected to the same vector in W

b. T, $\text{null} A^T = (\text{col} A)^\perp$ sym: $A^T = A \longrightarrow \text{null} A = (\text{col} A)^\perp$

c. F, A is not necessarily symmetric, $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, $\text{null} A = \text{col} A^\perp = \vec{0}$

d. T, if $\lambda = 0$, $\det(A - 0I) = 0$, $\det(A) = 0 \to$ not invertible

e. F, $\vec{0}$ is orthogonal to any $\vec{v}$, but $\{\vec{0}, \vec{v}\}$ is not lin. indpt.

f. F, $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \to$ not diagonalizable, but $A^3 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ which is diagonalizable.

g. T, if $A^3 = B^3$ and $A^3 = PDP^{-1}$, then there is a B where $B = PD^{1/3}P^{-1}$

h. T Columns are in the plane $x+y+z=0$ which is not the full $\mathbb{R}^3$, so not invertible and 0 is an eigenvalue

i. T $c_1 v_1 + \dots + c_n v_n = 0$ where $\{c_1 \dots c_n\} = 0$ in order to be lin. indpt

$\hookrightarrow v_i \cdot (c_1 v_1 + \dots + c_n v_n) = 0 \to c_i \|\vec{v}_i\|^2 = 0$, $\|\vec{v}_i\|^2 \neq 0$, c_i must be 0.

2) $A = \begin{bmatrix} -1 & -2 \\ 3 & 4 \end{bmatrix}$

a. $\begin{bmatrix} -1-\lambda & -2 \\ 3 & 4-\lambda \end{bmatrix}$ $\quad \chi_A(\lambda) = (-1-\lambda)(4-\lambda) + 6$

$0 = -4 - 3\lambda + \lambda^2 + 6 = \lambda^2 - 3\lambda + 2$

$\lambda = 2, 1$

$\lambda = 1$, $A_1 = \begin{bmatrix} -2 & -2 \\ 3 & 3 \end{bmatrix}$ $\quad \text{Nul}(A_1) = \text{Span}\left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$

$\lambda = 2$, $A_2 = \begin{bmatrix} -3 & -2 \\ 3 & 2 \end{bmatrix}$ $\quad \text{Nul}(A_2) = \text{Span}\left\{\begin{bmatrix} -2 \\ 3 \end{bmatrix}\right\}$

$P = \begin{bmatrix} -1 & -2 \\ 1 & 3 \end{bmatrix}$, $D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

b. $D^3 - 2D^2 + D$

$= \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$

$A^3 - 2A^2 + A \qquad A = PDP^{-1}$

$PD\cancel{P^{-1}}\cancel{P}D\cancel{P^{-1}}\cancel{P}DP^{-1} - 2 \cdot PD\cancel{P^{-1}}\cancel{P}DP^{-1} + PDP^{-1}$ $\qquad P^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$= PD^3P^{-1} - 2PD^2P^{-1} + PDP^{-1}$ $\qquad = -\frac{1}{1}\begin{bmatrix} 3 & 2 \\ -1 & -1 \end{bmatrix}$

$\begin{bmatrix} -1 & -2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 8 \end{bmatrix}\begin{bmatrix} -3 & -2 \\ 1 & 1 \end{bmatrix} - 2\begin{bmatrix} -1 & -2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} -3 & -2 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} -1 & -2 \\ 3 & 4 \end{bmatrix}$

$= \begin{bmatrix} -1 & -2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} -3 & -2 \\ 8 & 8 \end{bmatrix} - 2\begin{bmatrix} -1 & -2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} -3 & -2 \\ 4 & 4 \end{bmatrix} + \begin{bmatrix} -1 & -2 \\ 3 & 4 \end{bmatrix}$

$= \begin{bmatrix} -13 & -14 \\ 21 & 22 \end{bmatrix} - 2\begin{bmatrix} -5 & -6 \\ 9 & 10 \end{bmatrix} + \begin{bmatrix} -1 & -2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} -4 & -4 \\ 6 & 6 \end{bmatrix}$

can compute this first.

* $PD^3P^{-1} - 2PD^2P^{-1} + PDP^{-1} = P(D^3 - 2D^2)P^{-1} + D = P(D^3 - 2D^2 + D)P^{-1}$

distributive property can be used

3) $A = \begin{bmatrix} 3 & -4 & -4 \\ 2 & 1 & -4 \\ -2 & 0 & 5 \end{bmatrix}$ $\chi_A(\lambda) = -(\lambda-1)(\lambda-3)(\lambda-5)$

$\lambda = 1, 3, 5$

a. $\lambda = 1$, $\lambda = 3$, $\lambda = 5$

$A_1 = \begin{bmatrix} 2 & -4 & -4 \\ 2 & 0 & -4 \\ -2 & 0 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & -4 & -4 \\ 1 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$A_3 = \begin{bmatrix} 0 & -4 & -4 \\ 2 & -2 & -4 \\ -2 & 0 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & -2 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$A_5 = \begin{bmatrix} -2 & -4 & -4 \\ 2 & -4 & -4 \\ -2 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & -4 & -4 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$\text{Nul}(A_1) = \text{Span}\left\{\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}\right\}$ $\text{Nul}(A_3) = \text{Span}\left\{\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right\}$ $\text{Nul}(A_5) = \text{Span}\left\{\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}\right\}$

$\vec{v}_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ $\vec{v}_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ $\vec{v}_5 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

① $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$, $P = \begin{bmatrix} 2 & 1 & 0 \\ 0 & -1 & -1 \\ 1 & 1 & 1 \end{bmatrix}$

↳ switch columns of D, switch corresponding P col.

b. ② $D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $P = \begin{bmatrix} 1 & 0 & 2 \\ -1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ ④ $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, $P = \begin{bmatrix} 2 & 0 & 1 \\ 0 & -1 & -1 \\ 1 & 1 & 1 \end{bmatrix}$

③ $D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, $P = \begin{bmatrix} 0 & 2 & 1 \\ -1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}$ *P has infinite possibilities, while D has only 6. (There are ⑤ and ⑥ you can easily find.)

4) $\vec{u} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}$ where $W = \text{Span}\{\vec{u}, \vec{v}\}$

a. $T(x) = \text{proj}_W x = \dfrac{u \cdot x}{u \cdot u} u + \dfrac{v \cdot x}{v \cdot v} v$ $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$

$= \dfrac{x_1 + x_3 + x_4}{3} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} + \dfrac{x_1 + x_2 - x_3}{3} \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}$

$[T] = \frac{1}{3} \begin{bmatrix} 2x_1 + x_2 + 0 + x_4 \\ x_1 + x_2 - x_3 + 0 \\ 0 + x_2 + 2x_3 + x_4 \\ x_1 + 0 + x_3 + x_4 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2 & 1 & 0 & 1 \\ 1 & 1 & -1 & 0 \\ 0 & -1 & 2 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$

↳ not necessary to calculate $[T]$, think about the problem conceptually

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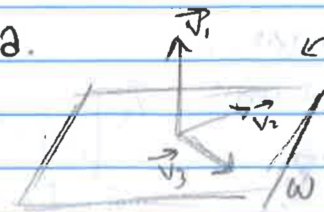
Midterm 2 Practice Cont.

a.



← example in $\mathbb{R}^3$

$\text{proj}_W \vec{v}_1 = \vec{0}$ → eigenvalue of 0

$\text{proj}_W \vec{v}_2 = \vec{v}_2$ → eigenvalue of 1

$\text{proj}_W \vec{v}_3 = \vec{v}_3$

Similar example can be found in $\mathbb{R}^4$, eigenvalues are 0 (w/ a multiplicity of 2), 1 (w/ a mult of 2)

b. w/ the same example in $\mathbb{R}^3$, you can see that there are three lin. independent eigenvectors that span $\mathbb{R}^3$, so [T] is diagonalizable. This can also be applied to the problem, but there would be 4 linearly independent vectors (2 for eigenvalue 0, 2 for eigenvalue 1).

5) $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} 3 \\ 1 \\ -1 \\ 0 \\ 2 \end{bmatrix}$ $\vec{v}_3 = \begin{bmatrix} 4 \\ -3 \\ 0 \\ 1 \\ 3 \end{bmatrix}$

a. $\vec{x}_1 = \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

$$\vec{x}_2 = v_2 - \text{proj}_{\vec{x}_1} \vec{v}_2 = \begin{bmatrix} 3 \\ 1 \\ -1 \\ 0 \\ 2 \end{bmatrix} - \frac{3+1-1+2}{5} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -2 \\ -1 \\ 1 \end{bmatrix}$$

$$\vec{x}_3 = v_3 - \text{proj}_{x_1} v_3 - \text{proj}_{x_2} v_3 = \begin{bmatrix} 4 \\ -3 \\ 0 \\ 1 \\ 3 \end{bmatrix} - 1 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{10}{10} \begin{bmatrix} 2 \\ 0 \\ -2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

orthogonal basis $= \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -4 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

b. method 1: normal equation

$$A\mathbf{x} = b$$

$$A^T A \mathbf{x} = A^T b$$

$$\mathbf{x} = (A^T A)^{-1} A^T b$$

$$\mathbf{x} = \left(\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 3 & 1 & -1 & 0 & 2 \\ 4 & -3 & 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 3 & 4 \\ 1 & 1 & -3 \\ 1 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 2 & 3 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 3 & 1 & -1 & 0 & 2 \\ 4 & -3 & 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \\ -1 \end{bmatrix}$$

$$= \left(\begin{bmatrix} 5 & 5 & 5 \\ 5 & 15 & 15 \\ 5 & 15 & 35 \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

method 2: orthogonal projection

↓ must be onto orthogonal basis

$$\vec{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad \vec{x}_2 = \begin{bmatrix} 2 \\ 0 \\ -2 \\ -1 \\ 1 \end{bmatrix}, \quad \vec{x}_3 = \begin{bmatrix} 1 \\ -4 \\ 1 \\ 1 \end{bmatrix}$$

$$\vec{b} = \begin{bmatrix} 0 \\ 1 \\ -1 \\ -1 \end{bmatrix}$$

↓ from a)

$$\operatorname{proj}_{\mathrm{Col}(A)} \vec{b} = \frac{b \cdot \vec{x}_1}{\vec{x}_1 \cdot \vec{x}_1} \vec{x}_1 + \frac{b \cdot \vec{x}_2}{\vec{x}_2 \cdot \vec{x}_2} \vec{x}_2 + \frac{b \cdot \vec{x}_3}{\vec{x}_3 \cdot \vec{x}_3} \vec{x}_3$$

$$= 0 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 2 \\ 0 \\ -2 \\ -1 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 1 \\ -4 \\ 1 \\ 1 \end{bmatrix}$$

$$X = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$