Midterm 1 Practice Problems

1) $x_1 + x_2 + cx_3 + x_4 = c$

$-x_2 + x_3 + 2x_4 = 0$

$x_1 + 2x_2 + x_3 - x_4 = -c$

$$\begin{bmatrix} 1 & 1 & c & 1 & c \\ 0 & -1 & 1 & 2 & 0 \\ 1 & 2 & 1 & -1 & -c \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & c & 1 & c \\ 0 & -1 & 1 & 2 & 0 \\ 0 & -1 & c-1 & 2 & 2c \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & c & 1 & c \\ 0 & -1 & 1 & 2 & 0 \\ 0 & 0 & 2-c & 0 & -2c \end{bmatrix} \qquad \left.\begin{aligned} 2-c &\neq 0 \\ c &\neq 2 \end{aligned}\right\}$$

any other c will make the last row have a pivot not in the last column

2) $T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$ sends $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ to $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ to $\begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$

a. $\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 5 \\ 0 & 2 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$

$$T\left(\begin{bmatrix} 5 \\ 6 \end{bmatrix}\right) = -1 \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 2 \cdot \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix}$$

b. $\begin{bmatrix} 1 & 3 & 7 \\ 2 & 4 & 8 \\ 3 & 5 & 7 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 7 \\ 0 & 2 & 6 \\ 0 & 4 & 14 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 7 \\ 0 & 1 & 3 \\ 0 & 0 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 7 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$

is $\begin{bmatrix} 7 \\ 8 \\ 7 \end{bmatrix}$ a linear combination of $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$?

no, no such $\vec{v}$ exists b/c $\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ is a basis of $\mathbb{R}^2$, so every (image) transformation should also be a linear combination of $T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) + T\left(\begin{bmatrix} 3 \\ 4 \end{bmatrix}\right)$.

3) $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 2 & 2 & 1 \end{bmatrix}$

method 1:

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 4 & 5 & 4 & 0 & 1 & 0 \\ 2 & 2 & 1 & 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 2 & 5 & 2 & 0 & -1 \\ 0 & 3 & 8 & 4 & -1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 3 & 2 & -1 & 1 \\ 0 & 0 & -1 & -2 & 2 & -3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -3 & -3 & 2 & -2 \\ 0 & 1 & 0 & -4 & 5 & -8 \\ 0 & 0 & 1 & 2 & -2 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 3 & -4 & 7 \\ 0 & 1 & 0 & -4 & 5 & -8 \\ 0 & 0 & 1 & 2 & -2 & 3 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 3 & -4 & 7 \\ -4 & 5 & -8 \\ 2 & -2 & 3 \end{bmatrix}$$

3) method 2:

use Cramer's rule

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 2 & 2 & 1 \end{bmatrix}^{-1} = \frac{1}{\det\left(\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 2 & 2 & 1 \end{bmatrix}\right)} \begin{bmatrix} 5\cdot1-2\cdot4 & -(4\cdot1-4\cdot2) & 4\cdot2-10 \\ -(2-6) & 1-6 & -(2-4) \\ 8-15 & -(4-12) & 5-8 \end{bmatrix}^T$$

$$\det\left(\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 2 & 2 & 1 \end{bmatrix}\right) = 1\cdot(5-8) - 2\cdot(4-8) + 3\cdot(8-10)$$
$$= -3+8-6 = -1$$

$$= -1 \cdot \begin{bmatrix} -3 & +4 & -7 \\ +4 & -5 & +8 \\ -2 & +2 & -3 \end{bmatrix} = \begin{bmatrix} 3 & -4 & 7 \\ -4 & 5 & -8 \\ 2 & -2 & 3 \end{bmatrix}$$

4) $T: \mathbb{R}^3 \to \mathbb{R}^3$ standard matrix w/ 2 pivots

$$T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \quad T\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$T(e_1+e_2+e_3) = 0 \longrightarrow$ b does not work

2pivots $\to$ should have 2 linearly independent columns $\to$ a doesn't work

5) $A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & 2 \end{bmatrix}$ $\quad B = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 2 & 1 & 0 & -1 \\ 0 & -1 & 2 & -1 \end{bmatrix}$

$$AB = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 2 & 1 & 0 & -1 \\ 0 & -1 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 & 1 \\ 4 & 1 & 2 & -1 \\ 2 & 1 & 0 & -1 \\ -1 & -1 & 3 & -3 \end{bmatrix}$$

b. $\det(AB) = \det(A)\cdot\det(B)$

$\because$ b/c # of columns of B > # of rows of B,
there is some nontrivial x where
$Bx = 0$

$\Downarrow$ there must also be a nontrivial x
where $ABx = 0$

$\Downarrow$ the columns of AB must be linearly dependent

$\det(AB) = 0$

Midterm 1 Practice cont.

6) $T: R^3 \to R^2$

$T(x,y,z) = (x-y+z, x)$

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$

pivots in every row, T is onto.

T is not one-to-one b/c there are not pivots in every column

Some y may be mapped from multiple x values in the domain.

7) $A = \begin{bmatrix} -2 & 1 \\ 1 & -1 \end{bmatrix}$ $T: R^2 \to R^2$ $B = \begin{bmatrix} 6 & 7 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}$ $S: R^2 \to R^4$

a. $ad - bc = 2 - 1 = 1$

$$A^{-1} = \frac{1}{1}\begin{bmatrix} -1 & -1 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ -1 & -2 \end{bmatrix} \longrightarrow \text{is invertible}$$

b. $S \cdot \underbrace{T \cdot T^{-1}}_{I} \cdot T$

$$\begin{bmatrix} 6 & 7 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -5 & -1 \\ 0 & 0 \\ -2 & 1 \\ 0 & 0 \end{bmatrix}$$

8) $$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 4 & 6 & 8 & 10 \\ 0 & 0 & 0 & -1 & 1 \\ 3 & 3 & 7 & 1 & 2 \\ 5 & -1 & 3 & 9 & 2 \end{bmatrix}$$

$$\det(A) = \cancel{0 \cdot \det\left(\begin{bmatrix} 1 & 1 & 1 & 1 \\ 4 & 6 & 8 & 10 \\ 3 & 7 & 1 & 2 \\ -1 & 3 & 9 & 2 \end{bmatrix}\right)} \rightarrow 0$$

$$\cancel{- 0 \cdot \det\left(\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 6 & 8 & 10 \\ 3 & 7 & 1 & 2 \\ 5 & 3 & 9 & 2 \end{bmatrix}\right)} \rightarrow 0$$

$$\cancel{+ 0 \cdot \det\left(\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 4 & 8 & 10 \\ 3 & 3 & 1 & 2 \\ 5 & -1 & 9 & 2 \end{bmatrix}\right)} \rightarrow 0$$

$$- (-1) \cdot \det\left(\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 4 & 6 & 10 \\ 3 & 3 & 7 & 2 \\ 5 & -1 & 3 & 2 \end{bmatrix}\right)$$

$$+ 1 \cdot \det\left(\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 4 & 6 & 8 \\ 3 & 3 & 7 & 1 \\ 5 & -1 & 3 & 9 \end{bmatrix}\right)$$

8) $\det(A) = \det\left(\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 4 & 6 & 10 \\ 3 & 3 & 7 & 2 \\ 5 & -1 & 3 & 2 \end{bmatrix}\right) + \det\left(\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 4 & 6 & 8 \\ 3 & 3 & 7 & 1 \\ 5 & -1 & 3 & 9 \end{bmatrix}\right)$

$\det\left(\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 4 & 6 & 10 \\ 3 & 3 & 7 & 2 \\ 5 & -1 & 3 & 2 \end{bmatrix}\right)$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 4 & 6 & 10 \\ 3 & 3 & 7 & 2 \\ 5 & -1 & 3 & 2 \end{bmatrix} \xrightarrow[\substack{R_3 \to R_3 - 3R_1 \\ R_4 \to R_4 - 5R_1}]{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 4 & 8 \\ 0 & 0 & 4 & -1 \\ 0 & -6 & -2 & -3 \end{bmatrix}$$

$$\xrightarrow{R_4 \to R_4 + 3R_2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 4 & 8 \\ 0 & 0 & 4 & -1 \\ 0 & 0 & 10 & 21 \end{bmatrix} \xrightarrow{\underbrace{R_3 \to \frac{1}{4}R_3}} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 4 & 8 \\ 0 & 0 & 1 & -\frac{1}{4} \\ 0 & 0 & 10 & 21 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 4 & 8 \\ 0 & 0 & 1 & -\frac{1}{4} \\ 0 & 0 & 0 & \frac{94}{4} \end{bmatrix}$$

$\frac{94}{2}$

$\frac{94}{2} \times 4 = \boxed{188}$

$\det\left(\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 4 & 6 & 8 \\ 3 & 3 & 7 & 1 \\ 5 & -1 & 3 & 9 \end{bmatrix}\right)$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 4 & 6 & 8 \\ 3 & 3 & 7 & 1 \\ 5 & -1 & 3 & 9 \end{bmatrix} \xrightarrow[\substack{R_3 \to R_3 - 3R_1 \\ R_4 \to R_4 - 5R_1}]{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 4 & 6 \\ 0 & 0 & 4 & -2 \\ 0 & -6 & -2 & 4 \end{bmatrix}$$

$$\xrightarrow{R_4 \to R_4 + 3R_2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 4 & 6 \\ 0 & 0 & 4 & -2 \\ 0 & 0 & 10 & 22 \end{bmatrix} \xrightarrow{\underbrace{R_3 \to \frac{1}{4}R_3}} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 4 & 6 \\ 0 & 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 10 & 22 \end{bmatrix} \xrightarrow{R_4 \to R_4 - 10R_3}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 4 & 6 \\ 0 & 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 & 27 \end{bmatrix} \; 54$$

$54 \times 4 = \boxed{216}$

$\det(A) = 188 + 216 = \boxed{404}$

Midterm 1 practice

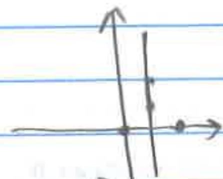
9) A transformation is linear if

i) $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(v)$

ii) $T(c\vec{u}) = cT(\vec{u})$

*check if the transformation sends $\begin{bmatrix}0\\0\end{bmatrix}$ to $\begin{bmatrix}0\\0\end{bmatrix}$

a.



← $\vec{0}$ is no longer $\vec{0}$

b.

← works ✓

c.

← works ✓

10) a. F, the echelon form is not unique, it can be tranformed into a diff. matrix using row operations.

b. F, $v_1 = e_1, v_2 = e_2, v_3 = e_3, v_4 = e_4, v_5 = \begin{bmatrix}1\\0\\1\\0\end{bmatrix}, v_6 = \begin{bmatrix}0\\1\\0\\1\end{bmatrix}$

Great example

c. F, $\begin{bmatrix}-1 & -1\\1 & 1\end{bmatrix}\begin{bmatrix}-1 & -1\\1 & 1\end{bmatrix} = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$

(2 det A = det 2A implies det A = 0 first of all)

d. F, A doesn't necessarily have to be 0, det (A) can be 0 without A being 0.

e. F, it means that some variable equals 0. would be inconsistent if the 1 was shifted over to the right.

f. F → $\vec{0}$ reflected across $y=1$ is no longer $\vec{0}$.

g. T, theorem 6.

h. T, can only have as many pivots as there are rows, must be a non-pivot column b/c $n>m$.

10) i. F, the set of columns of $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is linearly dependent.

j. F, has to contain the zero vector

k. F, $(AB)^{-1} = B^{-1}A^{-1}$

l. T, $A\vec{x} = \vec{e}_j$ being consistent for all $\vec{e}$ vectors means there is some $\vec{x}_j$ for $j = 1, 2, 3, 4$ that exists where $A\underbrace{\cancel{\vec{x}}}_{= \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \vec{x}_3 & \vec{x}_4 \end{bmatrix}} = I$, which means that A^{-1} exists.