Mathematics 54
Final Exam, 16 December 2019
180 minutes, 90 points

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INSTRUCTIONS:
Justify your answers, except when told otherwise.
All the work for a question should be on the respective sheet.

This is a CLOSED BOOK examination. Use of electronic devices (phones, tablets, etc) is NOT permitted. However, you may use a (two-sided) sheet of notes or formulas if you brought one with you.

Please turn in your finished examination **to your GSI** before leaving the room.

Credit: Q1:35; Q2:20; Q3:10; Q4:15; Q5:10. Bonus: 5

Question 1. (35 points) Select the correct answers, for 2.5 points each. No justification needed. Incorrect answers carry *no penalty* (but also no credit).
To receive credit, please record your choices by bubbling in the entries in the table on p.4. Mind how the answers are labeled (down columns).

1. When can we be certain that a system $A\mathbf{x} = \mathbf{b}$, with a 5×4 matrix A, is consistent?

$\mathrm{Col}\,A = (\mathrm{LNul}\,A)^{\perp}$

(a) Always
(b) When $\mathbf{b}$ is in $\mathrm{Nul}(A)$
(c) When $\mathbf{b} \perp \mathrm{Nul}(A)$
(d) When $\mathbf{b} \perp \mathrm{LNul}(A)$
(e) When A has four pivots
(f) When $\mathrm{Nul}(A) = \{\mathbf{0}\}$

2. For which vector $\mathbf{b}$ below does the system $\begin{bmatrix} 2 & 4 \\ 4 & 6 \\ 3 & 4 \end{bmatrix} \mathbf{x} = \mathbf{b}$ have a solution?

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(a) $\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$ (b) $\begin{bmatrix} 3 \\ 4 \\ 3 \end{bmatrix}$ (c) $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ (d) $\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$ (e) $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ (f) $\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$

3. For general matrix A, which of the following must remain *unchanged* under row operations?
(i) The row space (ii) The column space (iii) The positions of the pivot columns

Not (ii)
$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

(a) (i) and (ii)
(b) (ii) and (iii)
(c) (i) and (iii)
(d) (i), (ii) and (iii)
(e) (ii) but not (i) or (iii)
(f) All of them can change

4. Which of the following collections of vectors in $\mathbb{R}^4$ are linearly dependent?

Add up to zero.

(a) $\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_2 + \mathbf{e}_3, \mathbf{e}_3 + \mathbf{e}_4$
(b) $\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_2 + \mathbf{e}_3, \mathbf{e}_3 + \mathbf{e}_1$
(c) $\mathbf{e}_1 - \mathbf{e}_2, \mathbf{e}_2 - \mathbf{e}_3, \mathbf{e}_3 - \mathbf{e}_4$
(d) $\mathbf{e}_1 - \mathbf{e}_2, \mathbf{e}_2 - \mathbf{e}_3, \mathbf{e}_3 - \mathbf{e}_1$
(e) $\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1 - \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3$
(f) $\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_2 + \mathbf{e}_3, \mathbf{e}_3 + \mathbf{e}_4, \mathbf{e}_4$

5. Which of the matrices below have rank 2?

$$A = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 2 & 3 & 3 & 2 \\ 4 & 7 & 7 & 4 \end{bmatrix}; \quad B = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 1 \\ 0 & 1 & 0 \end{bmatrix}; \quad C = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 4 & 7 & 10 \end{bmatrix}; \quad D = \begin{bmatrix} 3 & 2 & 1 \\ 4 & 5 & 4 \\ 1 & 2 & 3 \end{bmatrix}$$

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(a) A, B and C but not D
(b) B and C but not A, D
(c) A and C but not B, D
(d) C and D but not A, B
(e) B, C and D but not A
(f) They all have rank 2

6. For a general $m \times n$ matrix A, the dimensions of $\mathrm{Col}(A)$ and of $\mathrm{Row}(A)$ agree *if and only if*

The rank theorem

(a) A is symmetric
(b) A is square
(c) A is diagonalizable
(d) A is invertible
(e) They always agree!
(f) A is orthogonal

7. If A and B are square matrices of the same size, we can safely conclude that

True even for A, B: non-square

(a) $AB = BA$
(b) $(A - B)(A + B) = A^2 - B^2$
(c) $AB^T = B^T A$
(d) $(AB)^T = B^T A^T$
(e) $(AB)^T = A^T B^T$
(f) None of the above.

8. If linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ satisfies $T(\mathbf{e}_1 - \mathbf{e}_2) = \mathbf{e}_2 - \mathbf{e}_3$, $T(\mathbf{e}_2 - \mathbf{e}_3) = \mathbf{e}_3 - \mathbf{e}_1$ and $T(\mathbf{e}_3 - \mathbf{e}_1) = \mathbf{e}_1 - \mathbf{e}_2$, then we can be certain that

Third condition just comes from first and second.

(a) T is invertible

(b) T is orthogonal

(c) $T(\mathbf{e}_1) = \mathbf{e}_2$

(d) $\mathbf{e}_1 - \mathbf{e}_2 + \mathbf{e}_3$ is in the range of T

● T has rank 2 or more

(f) T does not exist

9. The following is an eigenvalue of $A = \begin{bmatrix} -1 & 2 & 3 \\ 4 & 1 & 5 \\ 0 & 0 & 7 \end{bmatrix}$:

Cofactor using third row.

(a) 1 (b) 2 ● 3 (d) 4 (e) 5 (f) (-1)

10. Let A be a 3×4 matrix. Which of the following statements about $A^T A$ *cannot* be true?

$\mathrm{rk}\, A^TA = \mathrm{rk}\, A$

(a) It is square

(b) It is symmetric

● It is invertible

(d) It has rank 3

(e) It is diagonalizable over $\mathbf{R}$

(f) Its eigenvalues are ≥ 0

11. In which situation below can we be sure that the real $n \times n$ matrix A has *positive* determinant?

$\det A =$ product of eigenvalues

(a) A has positive entries

(b) There exists a matrix B with $AB = I_n$

(c) A has positive pivots

(d) A is diagonalizable

● All eigenvalues of A are positive real

(f) A is orthogonal

12. The least-squares solution to $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}$ is

$A^TAx = A^Tb$.

(a) $\mathbf{x} = 0$ (b) $\mathbf{x} = 1$ (c) $\mathbf{x} = 2$ (d) $\mathbf{x} = 3$ ● $\mathbf{x} = 4$ (f) Not listed

13. Pick the matrix below which is NOT diagonalizable:

(Mostly) look at the eigenvalues

(a) $\begin{bmatrix} 1 & 3 \\ 4 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ ● $\begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix}$ (e) $\begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix}$ (f) $\begin{bmatrix} 1 & 2 & 2 \\ 2 & 3 & 3 \\ 2 & 3 & 4 \end{bmatrix}$

14. The exponential of the matrix $\begin{bmatrix} 0 & -t \\ t & 0 \end{bmatrix}$ is

1) Should be real entries.

2) $t=0$ should be I_2.

(a) $\begin{bmatrix} 0 & e^{-t} \\ e^t & 0 \end{bmatrix}$

(b) $\begin{bmatrix} 0 & -e^t \\ e^t & 0 \end{bmatrix}$

● $\begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}$

(d) $\begin{bmatrix} \cos t & -i \sin t \\ i \sin t & \cos t \end{bmatrix}$

(e) $\begin{bmatrix} \cos t & i \sin t \\ i \sin t & \cos t \end{bmatrix}$

(f) $\begin{bmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{bmatrix}$

Question 2. (20 points)
Find a solution to the 2^{nd} order differential equation

$$x''(t) + x(t) = 4|t| \cdot \sin(t), \qquad t \in \mathbb{R}$$

with initial conditions $x(0) = x'(0) = 0$.
Check that your solution is twice differentiable everywhere, including at $t = 0$.
Is it three times differentiable there? Why or why not?
Use this to write down all the (twice differentiable) solutions of the equation.
Hint: Consider the cases $t \geq 0$ and $t \leq 0$ separately and use them to assemble a solution on $\mathbb{R}$.

Since the right hand side becomes different depending if $t \geq 0$ or $t < 0$, we find a solution separately: 1) $t \geq 0$ 2) $t < 0$. Note that the aux. eqns are both $r^2+1=0 \Rightarrow r = \pm i$ are sol'ns.
$x_1(t) = \cos t$, $x_2(t) = \sin t$ are homogeneous case solutions.

1) $x'' + x = 4t\sin t$

Try $x(t) = [(at+b)\cos t + (ct+d)\sin t] \times t = (at^2+bt)\cos t + (ct^2+dt)\sin t$.

$x'(t) = (2at + b + ct^2 + dt)\cos t + (-at^2 - bt + 2ct + d)\sin t$.

$x''(t) = (2ct + 2at + d - at^2 - bt + 2ct + d)\cos t + (-2at - b - ct^2 - dt - 2at - b + 2c)\sin t$.

So, $x''(t) + x(t) = (4ct + 2a + 2d)\cos t + (-4at - 2b + 2c)\sin t$.

$\Rightarrow 4c = 0$, $2a + 2d = 0$, $-4a = 4$, $-2b + 2c = 0$ $\Rightarrow a = -1$, $b, c = 0$. $d = 1$

$x(t) = -t^2\cos t + t\sin t$ $(t \geq 0)$

2) We actually try the same $x(t)$.

$\Rightarrow 4c = 0$, $2a + 2d = 0$, $-4a = -4$, $-2b + 2c = 0$ $\Rightarrow a = 1$, $b, c = 0$. and $d = -1$

$x(t) = t^2\cos t - t\sin t$ $(t < 0)$

Now, using $\underset{\cos t}{x_1(t)}$ and $\underset{\sin t}{x_2(t)}$, we need to modify each of $x(t)$ at $t \geq 0$ and $t < 0$ to satisfy $x(0) = 0$ and $x'(0) = 0$.

1 cont'd) new $x(t) = -t^2\cos t\ (+t\sin t) + a\cos t + b\sin t \Rightarrow x'(t) = -2t\cos t + t^2\sin t - a\sin t + b\cos t + \sin t + t\cos t$.

$t = 0 \Rightarrow$ $0 = 0 + a + 0$, $0 = 0 + 0 - 0 + b$. So, $a = b = 0$.

2 cont'd) new $x(t) = t^2\cos t - t\sin t + a\cos t + b\sin t$. Do the same thing and get $a = b = 0$ again!

Twice differentiable? Near $t = 0$, $x'(t)$ becomes $-2t\cos t + t^2\sin t + \sin t + t\cos t$ $(t > 0)$
$2t\cos t - t^2\sin t - \sin t - t\cos t$ $(t < 0)$.

They coincide at $t = 0$.

~~$x''(t)$ becomes $-2\cos t + 2t\sin t + 2t\sin t + t^2\cos t$~~

To check twice-differentiability, we only need to check $\lim \frac{x'(t) - 0}{t - 0}$ goes to the same number for $t \to 0^+$ and 0^-.

$= \lim \left(-2\cos t + t\sin t + \frac{\sin t}{t} + \cos t\right) = -2 + 0 + 1 + 1 = 0$ (if $t > 0$) ~~for both t~~

$\left(+2\cos t - t\sin t - \frac{\sin t}{t} - \cos t\right) = +2 + 0 - 1 - 1 = 0$. (if $t < 0$) They coincide.

$\boxed{\checkmark}$

You can check three-times differentiability in a similar way. It is.

All solutions: $f(t) = \begin{cases} -t^2\cos t + t\sin t + a\cos t + b\sin t & (t \geq 0) \\ t^2\cos t - t\sin t + a\cos t + b\sin t & (t < 0) \end{cases}$.

Question 3. (10 points)

Find the solution with initial condition $\mathbf{x}(0) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ for the ODE $\dfrac{d\mathbf{x}}{dt}(t) = \begin{bmatrix} 4 & 1 \\ -1 & 4 \end{bmatrix} \mathbf{x}(t)$.

Let's find the eigenvalues and eigenvectors.

Let $A = \begin{bmatrix} 4 & 1 \\ -1 & 4 \end{bmatrix}$ $\quad \chi_A(\lambda) = (\lambda-4)^2 + 1 = 0 \Rightarrow \lambda = 4 \pm i$.

$\mathrm{Nul}(A - (4+i)I) = \mathrm{Nul}\begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} = \mathrm{Span}\left\{\begin{bmatrix} 1 \\ i \end{bmatrix}\right\}$

$\mathrm{Nul}(A - (4-i)I) = \mathrm{Nul}\begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix} = \mathrm{Span}\left\{\begin{bmatrix} i \\ 1 \end{bmatrix}\right\}$

If $4+i = a+bi$ and $\begin{bmatrix} 1 \\ i \end{bmatrix} = \alpha + \beta i$, then the homogeneous case solutions are

$$e^{(a+bi)t}(\alpha+\beta i) = e^{at}\cdot(\cos bt + i\sin bt)(\alpha+\beta i)$$
$$= e^{at}\cdot(\cos bt\cdot\alpha - \sin bt\cdot\beta) + i\cdot e^{at}(\cos bt\cdot\beta + \sin bt\cdot\alpha).$$

So, $\mathbf{x}_1(t) = e^{4t}\left(\cos t\cdot\begin{bmatrix} 1 \\ 0 \end{bmatrix} - \sin t\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$ and $\mathbf{x}_2(t) = e^{4t}\left(\cos t\cdot\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \sin t\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$.

Now, for IVP, let $\mathbf{x}(t) = C_1\mathbf{x}_1(t) + C_2\mathbf{x}_2(t)$.

Then, $\mathbf{x}(0) = C_1\mathbf{x}_1(0) + C_2\mathbf{x}_2(0) = C_1\cdot e^0\cdot\left(\cos 0\begin{bmatrix} 1 \\ 0 \end{bmatrix} - 0\right) + C_2\cdot e^0\cdot\left(\cos 0\cdot\begin{bmatrix} 0 \\ 1 \end{bmatrix} + 0\right)$

$$= \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}. \quad \Rightarrow C_1 = C_2 = 2.$$

Therefore, $\mathbf{x}(t) = 2\mathbf{x}_1(t) + 2\mathbf{x}_2(t)$

$$= e^{4t}\begin{bmatrix} 2\cos t + 2\sin t \\ 2\cos t - 2\sin t \end{bmatrix} \quad \text{or} \quad e^{4t}\cos t\begin{bmatrix} 2 \\ 2 \end{bmatrix} + e^{4t}\sin t\begin{bmatrix} 2 \\ -2 \end{bmatrix}.$$

Question 4. (15 points)
Find a particular solution for the following vector-valued ODE:

$$\mathbf{x}'(t) = \begin{bmatrix} -5 & 2 \\ -6 & 2 \end{bmatrix} \cdot \mathbf{x}(t) + \frac{1}{e^{2t}+1}\begin{bmatrix} 5 \\ 8 \end{bmatrix}.$$

You may choose your method, but you must explain it briefly.
Help with integrals: $\int \frac{dt}{t^2+1} = \arctan(t) + C$

Use eigenvector method: First, let $A = \begin{bmatrix} -5 & 2 \\ -6 & 2 \end{bmatrix}$. $\chi_A(\lambda) = (\lambda+5)(\lambda-2)+12$
$= \lambda^2+3\lambda+2.$

$\lambda_1 = -2 \Rightarrow \text{Nul}\begin{bmatrix} -3 & 2 \\ -6 & 4 \end{bmatrix} \ni \begin{bmatrix} 2 \\ 3 \end{bmatrix} = v_1$

$\lambda_2 = -1 \Rightarrow \text{Nul}\begin{bmatrix} -4 & 2 \\ -6 & 3 \end{bmatrix} \ni \begin{bmatrix} 1 \\ 2 \end{bmatrix} = v_2.$

Note that $\begin{bmatrix} 5 \\ 8 \end{bmatrix} = 2\cdot\begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ (You can easily solve and find this.)

Now, let $X(t) = \xi_1(t)\, v_1 + \xi_2(t)\, v_2.$

$(LHS) = \xi_1'(t) v_1 + \xi_2'(t) v_2$ $\quad (RHS) = \underbrace{A\xi_1(t)v_1}_{\xi_1(t)(-2)v_1} + \underbrace{A\xi_2(t)v_2}_{\xi_2(t)\cdot(-1)v_2} + \frac{1}{e^{2t}+1}(2v_1+v_2)$

Hence, we get $\xi_1'(t) = -2\xi_1(t) + 2\cdot\frac{1}{e^{2t}+1}$ and $\xi_2'(t) = -\xi_2(t) + \frac{1}{e^{2t}+1}.$

Using the integrating factor, we get, $\xi_1(t) = \frac{\int \frac{e^{2t}}{e^{2t}+1}dt}{e^{2t}}$ and $\xi_2(t) = \frac{\int \frac{e^{t}}{e^{2t}+1}dt}{e^{t}}.$

$\int \frac{e^{2t}}{e^{2t}+1}dt = \frac{1}{2}\ln(e^{2t}+1)$ (b/c $\int \frac{f'}{f} = \ln f$)

$\int \frac{e^t}{e^{2t}+1}dt = \int \frac{s}{s^2+1}\cdot\frac{1}{s}ds = \int \frac{1}{s^2+1}ds = \arctan(s) = \arctan(e^t)$

$\left(s = e^t \text{ and } ds = e^t dt \Rightarrow \frac{1}{s}ds = dt\right)$

Therefore, $X(t) = \frac{\ln(e^{2t}+1)}{2e^{2t}}\begin{bmatrix} 2 \\ 3 \end{bmatrix} + \frac{\arctan(e^t)}{e^t}\cdot\begin{bmatrix} 1 \\ 2 \end{bmatrix}.$

Question 5. (10 points)
Find all the numbers λ for which the differential equation $x''(t) = \lambda x(t)$ has *non-zero* solutions $x(t)$ which satisfy $x(0) = x(\pi) = 0$. For each such λ, write down all such solutions.
Suggestion: Write the general solution of the equation for a fixed λ, and adjust the constants to make $x(0), x(\pi)$ vanish. You may assume that λ is real, if it helps your calculation.

The auxiliary equation is $r^2 - \lambda = 0$. Let μ and $-\mu$ be two answers of the equation. ~~(If $\lambda=0$, they could be the same)~~ (We will do $\lambda=0$ case separately.)

Then $x(t) = C_1 e^{\mu t} + C_2 e^{-\mu t}$.

$x(0) = C_1 + C_2 = 0$ We need non-zero solutions, so the

$x(\pi) = C_1 \cdot e^{\mu\pi} + C_2 \cdot e^{-\mu\pi} = 0$. equation has nontrivial C_1 and C_2.

In other words, $\begin{bmatrix} 1 & 1 \\ e^{\mu\pi} & e^{-\mu\pi} \end{bmatrix}$ should not be invertible. $\Rightarrow e^{-\mu\pi} - e^{\mu\pi} = 0$.

$\Rightarrow 1 - e^{2\mu\pi} = 0$.

Now, suppose μ can be written as $a + bi$. Then, $e^{2\mu\pi} = e^{2a\pi + 2b\pi i}$
$= e^{2a\pi} \cdot (\cos 2b\pi + i \sin 2b\pi)$

For this to be 1, $\sin 2b\pi = 0$ and $\cos 2b\pi = 1$ and $e^{2a\pi} = 1$.

$\Rightarrow 2b\pi$ = multiple of 2π and $2a\pi = 0$ $\Rightarrow a = 0$, b: an integer.

Therefore, $\lambda = \mu^2 = (\underset{b:\text{ integer}}{bi})^2 = -1, -4, -9, -25, \cdots$ (we are doing $\lambda = 0$ case soon.)

If $\lambda = 0$, $\Rightarrow x''(t) = 0$ has solutions 1 and t. $x(t) = C_1 + C_2 t$. By plugging $t = 0$ and π, we get $x(t) = 0$, so this is not the case we are looking for.

Conclusion: $\lambda = -1, -4, -9, -25, \cdots$ ($-n^2$ for $n (\neq 0)$ integer)

For each n ($\lambda = -n^2$), the auxiliary equation has $r = \pm ni$ as roots.

$x(t) = C_1 \cos nt + C_2 \sin nt$. $x(0) = 0 \Rightarrow C_1 = 0$
$x(\pi) = 0 \Rightarrow C_1 = 0$.

So, $x(t) = C \cdot \sin nt$ ($C \neq 0$) for $\lambda = -n^2$ (n: nonzero integer)

Short remark: You may assume λ: real and divide it into two cases (or three) $\lambda < 0$, $\lambda = 0$, $\lambda > 0$. and do a similar thing. Then, you don't need any technical things.

Bonus Question. (5 points)
You can only get credit for this if you solved Q5 correctly.
(a) For any two twice-differentiable functions f, g which vanish at 0 and at π, show that

$$\int_0^\pi f''(t)g(t)dt = \int_0^\pi f(t)g''(t)dt.$$

(b) By using (a), or by direct computation, show that two solutions f, g as in Q5, but associated to two *different* values of λ are *orthogonal* in the sense that

$$\int_0^\pi f(t)g(t)dt = 0.$$

Cultural comment: This, plus Q5, show that the eigenfunctions of the second derivative operator, $f \mapsto f''$, form an orthogonal collection in the space of differentiable functions on $[0, \pi]$ vanishing at the endpoints. General theorems ensure that it is *complete*, so that any function above has a series expansion, convergent in mean square, in terms of the eigenfunctions you found in Q5.

(a) Integration by parts: $\int_0^\pi f''g = f'g\Big|_0^\pi - \int_0^\pi f'g' = 0 - 0 - \int_0^\pi f'g'$

f or g are 0 at ~~0~~ 0, π } the same.

$\int_0^\pi fg'' = f\cdot g'\Big|_0^\pi - \int_0^\pi f'g' = 0 - 0 - \int_0^\pi f'g'$

(b) Let f be a function from Q5 w/ $\lambda = \lambda_f$
" g " " " $\lambda = \lambda_g$ and $\lambda_f \neq \lambda_g$.

Then, $f'' = \lambda_f f$ and $g'' = \lambda_g \cdot g$. So,

$$\int_0^\pi f''g = \int_0^\pi \lambda_f \cdot f \cdot g = \lambda_f \int_0^\pi f \cdot g$$

$$\int_0^\pi f \cdot g'' = \int_0^\pi f \cdot \lambda_g \cdot g = \lambda_g \int_0^\pi fg$$

But these two are the same because f and g are ~~the~~ such functions which vanish at 0 and π (See Q5.).

Therefore, $\lambda_f \cdot \int_0^\pi fg = \lambda_g \cdot \int_0^\pi fg$ but $\lambda_f \neq \lambda_g$, so $\int_0^\pi f(t)g(t)dt = 0$.