1. Mark "T" if the statement is always true, "F" if it is sometimes false. ***No** explanations are needed.*

1) T F | If all of the eigenvalues of a matrix are not real but complex, then it must be invertible.

2) T F | If A is an $n \times n$ orthogonal matrix then the RREF of A must have n pivots.

3) T F | If λ is an eigenvalue of A then λ^2 must be an eigenvalue of A^2.

4) T F | The set of diagonalizable 2×2 matrices is a subspace of the vector space $M_{2\times 2}$.

5) T F | If the 3×3 matrix A has two rows that are the same, then $\det A = 0$.

6) T F | Let A be an $n \times n$ matrix. If A^9 is the zero matrix, then the only eigenvalue of A is 0.

7) T F | If A is a square matrix and $A^5 = I$ then A is invertible.

8) T F | If A is a 5×5 matrix such that $\det(2A) = \det A$ then $A = 0$.

9) T F | If T is a one-to-one linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$ then T is onto.

10) T F | Let A be an $n \times n$ matrix such that $A^2 = A$, then A is invertible.

11) T F | Every symmetric $n \times n$ matrix with real entries is similar to a diagonal matrix with real entries.

2. Select the correct answers. Be aware that there might be more than one answer to each problem.

1) The exponential of a square matrix A is
   (a) The sum of the series $I + A + A^2 + A^3 + \cdots$
   (b) The sum of the series $I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots$
   (c) The matrix whose (i, j)-entry is $\exp(A_{ij})$, where A_{ij} is the (i, j)-entry of A
   (d) The diagonal matrix with entries e^{λ_i}, where the λ_i are the eigenvalues of A

2) The exponential of the matrix $\begin{bmatrix} t & t \\ 0 & -t \end{bmatrix}$ is
   (a) $\begin{bmatrix} e^t & e^t \\ t & e^{-t} \end{bmatrix}$ (b) $\begin{bmatrix} e^t & te^t \\ 0 & e^{-t} \end{bmatrix}$
   (c) $\begin{bmatrix} e^t & (e^t - e^{-t})/2 \\ 0 & e^{-t} \end{bmatrix}$ (d) $\begin{bmatrix} e^t & (e^t + e^{-t})/2 \\ 0 & e^{-t} \end{bmatrix}$

3) Pick the matrix on the list which is NOT diagonalizable over $\mathbb{C}$, if any; else, pick option (e).
   (a) $\begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$
   (d) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$ (e) All of them are diagonalizable over $\mathbb{C}$.

4) If A and B are matrices such that $AB = 0$, we can safely conclude that
   (a) $\operatorname{Nul} A$ contains $\operatorname{Nul} B$ (b) $BA = 0$
   (c) $\operatorname{Nul} A$ contains $\operatorname{Col} B$ (d) $\operatorname{Col} A$ contains $\operatorname{Nul} B$

5) Which subspace of $\mathbb{R}^4$ is the orthogonal complement of the subspace defined by the conditions

$$\{[x_1, x_2, x_3, x_4]^T : x_1 + x_3 = 0 \text{ and } x_1 - x_2 + x_3 - x_4 = 0\}?$$

   (a) $\operatorname{Span}([1, 0, 1, 0]^T, [1, -1, 1, -1]^T)$ (b) $\operatorname{Span}([1, 2, -1, -2]^T, [1, 1, 1, 1]^T)$
   (c) $\operatorname{Span}([1, 2, -1, 2]^T, [1, 1, -1, -1]^T)$ (d) $\operatorname{Span}([0, 1, 0, 1]^T, [1, 1, 1, 1]^T)$

6) Which linear transformation T has the image not of dimension 2?
   (a) $T : \mathbb{R}^3 \to \mathbb{R}^4$ sending $[x, y, z]^T$ to $[x, y, 0, 0]^T$
   (b) $T : \mathbb{P}_2 \to \mathbb{P}_2$ sending $at^2 + bt + c$ to $2at + b - 2c$
   (c) $T : \mathbb{P}_3 \to \mathbb{P}_3$ sending $f(t)$ to $f''(t)$
   (d) The orthogonal projection map from $\mathbb{R}^3$ to the plane defined by $x - 2y + 3z = 0$
   (e) All of them have 2-dimensional images.

3. Consider the 2×2 matrix

$$M_a = \begin{bmatrix} a & 2-a \\ 2+a & -a \end{bmatrix}.$$

a) Find all real values of a such that M_a is invertible.

b) Find all real values of a such that M_a is diagonalizable.

c) Find all eigenvectors of M_1.

4. Given a linear second-order equation

$$y''(t) + ay'(t) + by(t) = f(t),$$

only information you have is a set of three solutions to the equation. They are

$$t + e^t \cos 2t + e^{2t} \sin t,\ \ t + e^{2t} \sin t,\ \ t + e^t \sin 2t + e^{2t} \sin t.$$

Find a, b, and $f(t)$.

5. Solve the following initial value problems:

   a) $y'' + y' = t^2$ with $y(0) = 0$ and $y'(0) = 0$.

   b) $y'' + y = \sec t$ with $y(0) = 0$ and $y'(0) = 0$.

   c) $\mathbf{x}(t)' = A\mathbf{x}(t) + \mathbf{f}(t)$ where $A = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}$ and $\mathbf{f}(t) = \begin{bmatrix} 0 \\ 4e^t \end{bmatrix}$ with $\mathbf{x}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

6. Consider the following differential equation in normal form:

$$\mathbf{x}(t)' = A\mathbf{x}(t) + \begin{bmatrix} 1 \\ 2 \end{bmatrix} t^{-1}e^{3t} \text{ where } A = \begin{bmatrix} -1 & 2 \\ -6 & 6 \end{bmatrix}.$$

   a) Find a fundamental matrix of the corresponding homogeneous equation.
   b) Compute e^{At} using (a).
   c) Find a particular solution using *eigenvector method*.[1]

[1] That is, for v_1 and v_2 eigenvectors composing a basis, set $\mathbf{x}_p(t) = \xi_1(t)v_1 + \xi_2(t)v_2$ and solve for $\xi_1(t)$ and $\xi_2(t)$.

7. (Extra) Let A be a 2×2 matrix such that[2]

$$A \begin{bmatrix} -1 \\ 4 \end{bmatrix} = -4 \begin{bmatrix} -1 \\ 4 \end{bmatrix}, \ A \begin{bmatrix} 0 \\ 1 \end{bmatrix} = -4 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 4 \end{bmatrix}.$$

Find the functions $x(t)$ and $y(t)$ with initial values $x(0) = -2$, $y(0) = 11$ that satisfy the system of differential equations

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}' = A \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}.$$

[2]Hint. This tells you that $A = PDP^{-1}$ where $P = \begin{bmatrix} -1 & 0 \\ 4 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} -4 & 1 \\ 0 & -4 \end{bmatrix}$.