Answer Key

Linear Algebra
Differential Equations Math 54 Lecture 002 Practice Final Exam [LA] Nov 22, 2019

1. Mark "T" if the statement is always true, "F" if it is sometimes false. *No explanations are needed.*

T F | If the span of $v_1, \cdots, v_n$ contains w_1, w_2, w_3, it contains the span of these three vectors.

T F | If B is an $n \times 5$ matrix, the set of matrices $A \in M_{m\times n}$ such that $AB = 0$ is a subspace of $M_{m\times n}$.

T F | If A and B are $n \times n$ square matrices then $(A+B)^2 = A^2 + 2AB + B^2$.

T F | There exists a set of three nonzero orthogonal vectors in $\mathbb{R}^2$.

T F | There exists a real 3×3 matrix A such that $\dim \mathrm{Col} A = \dim \mathrm{Nul} A$.

T F | There exists a real 3×3 orthogonal matrix U such that $\det U = 2$.

T F | If A and B are $n \times n$ matrices, and AB is invertible, then A and B are invertible.

T F | If a matrix A has linearly dependent columns then $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions.

T F | A least-squares solution $\hat{\mathbf{x}}$ of a linear system $A\mathbf{x} = \mathbf{b}$, with A of size $m \times n$, is always characterized by the following: $\hat{\mathbf{x}}$ is in Col$A$ and $||A\hat{\mathbf{x}} - \mathbf{b}||$ is as short as possible.

T F | The characteristic polynomial of a 2×2 matrix A is $\lambda^2 + \lambda \operatorname{Tr} A + \det A$.

T F | If $T : \mathbb{R}^n \to \mathbb{R}^m$ is an one-to-one linear transformation, then $n < m$.

T F | Let v, w, z be vectors in $\mathbb{R}^n$. If $v \cdot w$ and $v \cdot z$, then $w \cdot z = 0$.

T F | If $v \cdot z = w \cdot z$ for all $z \in \mathbb{R}^n$, then $v = w$.

2. Let A be a 3×3 matrix with eigenvalues 0, 1, 2 and corresponding eigenvectors v_0, v_1, and v_2.

   a. Find bases for Col$A$ and Nul$A$.

   b. Find two different vectors $\mathbf{x}$ such that $A\mathbf{x} = 2v_1 + v_2$.

a. v_0, v_1, v_2 are linearly independent vectors b/c they correspond to distinct eigenvalues.
↳ 3 lin. independent vectors make a basis that spans $\mathbb{R}^3$
↳ any $v \in \mathbb{R}^3$ is a linear combo of v_0, v_1, v_2 $\vec{v} = C_0 v_0 + C_1 v_1 + C_2 v_2$.

$\text{Col}\,A = \{A\vec{v}\} = \{A(c_0 v_0 + c_1 v_1 + c_2 v_2)\} = \{c_0 Av_0 + c_1 Av_1 + c_2 Av_2\}$
$= \{0 + c_1 v_1 + 2c_2 v_2\}$
$= \text{Span}\{v_1, v_2\}$

$\text{Nul}\,A = \{v \in \mathbb{R}^3, Av = 0\} = \{c_0 v_0 + c_1 v_1 + c_2 v_2\} \Rightarrow c_1, c_2$ must equal 0
$= \{c_0 v_0\} = \text{Span}\{v_0\}$

b. $Av = A(c_0 v_0 + c_1 v_1 + c_2 v_2) = c_1 v_1 + 2c_2 v_2 = 2v_1 + v_2$
$c_1 = 2,\ c_2 = \frac{1}{2}$, while c_0 can equal anything
$\vec{x}_1 = 2\vec{v}_1 + \frac{1}{2}\vec{v}_2$
$\vec{x}_2 = \vec{v}_0 + 2\vec{v}_1 + \frac{1}{2}\vec{v}_2$

3. Let $\mathbb{P}_2$ be the vector space of polynomials of degree at most 2. Let $\mathcal{B} = \{1, x, x^2\}$ be a basis for $\mathbb{P}_2$ and $T : \mathbb{P}_2 \to \mathbb{P}_2$ denote the mapping sending f to $f' + f$. Find the matrix A for T with respect to the basis $\mathcal{B}$ and find the eigenvectors and eigenvalues of A.

$A = \begin{bmatrix} | & | & | \\ T(e_1) & T(e_2) & T(e_3) \\ | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \Leftarrow$

$f \to f' + f$

$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \to 1$ $\quad 1' + 1 = 0 + 1 \quad T(e_1) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \to x$ $\quad x' + x = 1 + x \quad T(e_2) = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \to x^2$ $\quad x^{2\prime} + x^2 = 2x + x^2 \quad T(e_3) = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$

$A - \lambda I = \begin{bmatrix} 1-\lambda & 1 & 0 \\ 0 & 1-\lambda & 2 \\ 0 & 0 & 1-\lambda \end{bmatrix}.$

$\chi(\lambda) = (1-\lambda)[(1-\lambda)^2 - 2\cdot 0] = (1-\lambda)^3$

$\lambda = 1$, with multiplicity of 3

$A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad x_1 = x_1,\ x_2 = 0,\ x_3 = 0 \quad \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

4. Let $v = \begin{bmatrix}1 & -1 & -1 & 1\end{bmatrix}^T$ and $w = \begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix}^T$. For $\mathbf{x} \in \mathbb{R}^4$, let $T(\mathbf{x}) = (\mathbf{x}\cdot v)v + (\mathbf{x}\cdot w)w$.

   a. Show that T is a linear transformation from $\mathbb{R}^4$ to $\mathbb{R}^4$.

   b. Find two eigenvectors of T and one non-zero vector $\mathbf{x}$ such that $T(\mathbf{x}) = 0$.

a.

$$T(\vec{x})+T(\vec{y}) = (\vec{x}\cdot\vec{v})\vec{v} + (\vec{y}\cdot\vec{v})\vec{v} + (\vec{y}\cdot\vec{w})\vec{w} + (\vec{x}\cdot\vec{w})\vec{w}$$
$$= [(\vec{x}+\vec{y})\cdot\vec{v}]\vec{v} + [(\vec{x}+\vec{y})\cdot\vec{w}]\vec{w} = T(\vec{x}+\vec{y})$$
$$T(c\vec{x}) = (c\vec{x}\cdot\vec{v})\vec{v} + (c\vec{x}\cdot\vec{w})\vec{w}$$
$$= c(\vec{x}\cdot\vec{v})\vec{v} + c(\vec{x}\cdot\vec{w})\vec{w} = c((\vec{x}\cdot\vec{v})\vec{v} + (\vec{x}\cdot\vec{w})\vec{w}) = cT(\vec{x})$$

b. $A = \begin{bmatrix} T(e_1) & T(e_2) & T(e_3) & T(e_4)\end{bmatrix}$

$$T\left(\begin{bmatrix}1\\0\\0\\0\end{bmatrix}\right) = \left(\begin{bmatrix}1\\0\\0\\0\end{bmatrix}\cdot\begin{bmatrix}1\\-1\\-1\\1\end{bmatrix}\right)\begin{bmatrix}1\\-1\\-1\\1\end{bmatrix} + \left(\begin{bmatrix}1\\0\\0\\0\end{bmatrix}\cdot\begin{bmatrix}1\\1\\1\\1\end{bmatrix}\right)\begin{bmatrix}1\\1\\1\\1\end{bmatrix} = \begin{bmatrix}2\\0\\0\\2\end{bmatrix}$$

$$T\left(\begin{bmatrix}0\\1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}0\\2\\2\\0\end{bmatrix} \quad T\left(\begin{bmatrix}0\\0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}0\\2\\2\\0\end{bmatrix} \quad T\left(\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}2\\0\\0\\2\end{bmatrix}$$

$$= \begin{bmatrix}2&0&0&2\\0&2&2&0\\0&2&2&0\\2&0&0&2\end{bmatrix}$$

Find eigenvectors/eigenvalues w/ 2 methods:

① $A-\lambda I = \begin{bmatrix}2-\lambda&0&0&2\\0&2-\lambda&2&0\\0&2&2-\lambda&0\\2&0&0&2-\lambda\end{bmatrix}$

② $T(av+bw) = ((av+bw)\cdot v)v + ((av+bw)\cdot w)w$ You do not need matrix computation here.

$$= 4av+4bw$$
$\lambda(av+bw)$

$\lambda av + \lambda bw = 4av+4bw$, $\lambda = 4$

to find nonzero vector,

$\mathbf{x}\cdot v + \mathbf{x}\cdot w = 0$, same as finding $\text{Nul}\begin{bmatrix}1&-1&-1&1\\1&1&1&1\end{bmatrix} \Rightarrow \begin{bmatrix}1&0&0&1\\0&1&1&0\end{bmatrix}$

any vector in $\text{Span}\left\{\begin{bmatrix}0\\-1\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\0\\1\end{bmatrix}\right\}$ works.

5. Find the change-of-coordinates matrix from the basis $\mathcal{B} = \{1, (1+t)^2, (1-t)^2, t^3\}$ of $\mathbb{P}_3$ to the standard basis $\mathcal{C} = \{1, t, t^2, t^3\}$.

$$M[\mathbf{x}]_B = [\mathbf{x}]_C$$
$$M = \begin{bmatrix}[b_1]_C & [b_2]_C & [b_3]_C & [b_4]_C\end{bmatrix}$$

$b_1 = 1\cdot 1 + 0\cdot t + 0\cdot t^2 + 0\cdot t^3$

$b_2 = 1\cdot 1 + 2\cdot t + 1\cdot t^2 + 0\cdot t^3$

$b_3 = 1\cdot 1 + -2\cdot t + 1\cdot t^2 + 0\cdot t^3$

$b_4 = 0\cdot 1 + 0\cdot t + 0\cdot t^2 + 1\cdot t^3$

$\rightarrow$ $[b_1]_C = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}$ $[b_3]_C = \begin{bmatrix}1\\-2\\1\\0\end{bmatrix}$ $[b_2]_C = \begin{bmatrix}1\\2\\1\\0\end{bmatrix}$ $[b_4]_C = \begin{bmatrix}0\\0\\0\\1\end{bmatrix}$

$$M = \begin{bmatrix}1&1&1&0\\0&2&-2&0\\0&1&1&0\\0&0&0&1\end{bmatrix}$$

6. Let $M_{2\times 2}$ be the vector space of 2×2 real matrices. Let

$$H = \{X \in M_{2\times 2} : X \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}\}.$$

a. Prove that H is a subspace.

b. Find a basis for H and compute $\dim H$.

a. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ → includes zero

$X\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $Y\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ → closed under addition

$(X+Y)\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

↳ distributive property

$(cX)\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ → closed under scalar multiplication

$cX\begin{bmatrix} 1 \\ -1 \end{bmatrix} = c\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Note that

b. A basis for every $M_{2\times 2}$ matrix is $\left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$ w/ a dimension of 4.

Now, to find what H looks like,

let $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in H$.

Then, $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$a - b = 0$, $a = b$
$c - d = 0$, $c = d$ → restricts the basis to $\left\{\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}\right\}$ w/ a dimension of 2.