1. Mark "T" if the statement is always true, "F" if it is sometimes false. ***No** explanations are needed.*

T F | If the span of $v_1, \cdots, v_n$ contains w_1, w_2, w_3, it contains the span of these three vectors.

T F | If B is an $n \times 5$ matrix, the set of matrices $A \in M_{m\times n}$ such that $AB = 0$ is a subspace of $M_{m\times n}$.

T F | If A and B are $n \times n$ square matrices then $(A + B)^2 = A^2 + 2AB + B^2$.

T F | There exists a set of three nonzero orthogonal vectors in $\mathbb{R}^2$.

T F | There exists a real 3×3 matrix A such that $\dim \mathrm{Col}A = \dim \mathrm{Nul}A$.

T F | There exists a real 3×3 orthogonal matrix U such that $\det U = 2$.

T F | If A and B are $n \times n$ matrices, and AB is invertible, then A and B are invertible.

T F | If a matrix A has linearly dependent columns then $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions.

T F | A least-squares solution $\hat{\mathbf{x}}$ of a linear system $A\mathbf{x} = \mathbf{b}$, with A of size $m \times n$, is always characterized by the following: $\hat{\mathbf{x}}$ is in $\mathrm{Col}A$ and $||A\hat{\mathbf{x}} - \mathbf{b}||$ is as short as possible.

T F | The characteristic polynomial of a 2×2 matrix A is $\lambda^2 + \lambda \,\mathrm{Tr}\, A + \det A$.

T F | If $T : \mathbb{R}^n \to \mathbb{R}^m$ is an one-to-one linear transformation, then $n < m$.

T F | Let v, w, z be vectors in $\mathbb{R}^n$. If $v \cdot w$ and $v \cdot z$, then $w \cdot z = 0$.

T F | If $v \cdot z = w \cdot z$ for all $z \in \mathbb{R}^n$, then $v = w$.

2. Let A be a 3×3 matrix with eigenvalues 0, 1, 2 and corresponding eigenvectors v_0, v_1, and v_2.
   a. Find bases for Col$A$ and Nul$A$.
   b. Find two different vectors $\mathbf{x}$ such that $A\mathbf{x} = 2v_1 + v_2$.

3. Let $\mathbb{P}_2$ be the vector space of polynomials of degree at most 2. Let $\mathcal{B} = \{1, x, x^2\}$ be a basis for $\mathbb{P}_2$ and $T : \mathbb{P}_2 \to \mathbb{P}_2$ denote the mapping sending f to $f' + f$. Find the matrix A for T with respect to the basis $\mathcal{B}$ and find the eigenvectors and eigenvalues of A.

4. Let $v = \begin{bmatrix} 1 & -1 & -1 & 1 \end{bmatrix}^T$ and $w = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^T$. For $\mathbf{x} \in \mathbb{R}^4$, let $T(\mathbf{x}) = (\mathbf{x} \cdot v)v + (\mathbf{x} \cdot w)w$.

   a. Show that T is a linear transformation from $\mathbb{R}^4$ to $\mathbb{R}^4$.

   b. Find two eigenvectors of T and one non-zero vector $\mathbf{x}$ such that $T(\mathbf{x}) = 0$.

5. Find the change-of-coordinates matrix from the basis $\mathcal{B} = \{1, (1+t)^2, (1-t)^2, t^3\}$ of $\mathbb{P}_3$ to the standard basis $\mathcal{C} = \{1, t, t^2, t^3\}$.

6. Let $M_{2\times 2}$ be the vector space of 2×2 real matrices. Let

$$H = \{X \in M_{2\times 2} : X \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}\}.$$

   a. Prove that H is a subspace.
   b. Find a basis for H and compute $\dim H$.