1. Suppose that B is an invertible square matrix with the property that for both B and B^{-1}, all of their entries are integers. Show that $\det B$ is 1 or -1.

If you recall the way to find the determinant of a matrix, then you take products or sums of entries. There is no dividing operation applied. Hence, if B has only integer entries, $\det B$ is also an integer. Since B^{-1} is also assumed to have integer entries, $\det B^{-1}$ is also an integer. However, $\det B \cdot \det B^{-1}$ is $\det(BB^{-1}) = \det I = 1$ always. The only integers multiplied up to 1 are $1 \cdot 1$ or $-1 \cdot -1$ $\blacksquare$

2. Let B be an $n \times n$ matrix satisfying $B^T = -B$.[1] By considering the determinant, show that B is not invertible if n is odd.

Suppose that n is odd. Take the determinant to $B^T = -B$. On the left hand side, we get $\det B^T$ which is the same as $\det B$. On the right hand side, we have $\det(-B) = (-1)^n \cdot \det B = -\det B$ b/c n is odd. Therefore, we get

$$\det B = -\det B.$$

So, $\det B = 0$ which means that B is not invertible. $\blacksquare$

3. Let

$$A = \begin{bmatrix} 54 & 81 \\ -9 & 0 \end{bmatrix}.$$

a. Find the eigenvalues of A and the corresponding eigenvector.

b. Let D be the diagonal matrix whose diagonal entries are exactly the eigenvalues from the above. Check that N which is defined to be $A - D$ satisfies $N^2 = 0$.[2]

c. Use $N^2 = 0$ to find a matrix B such that $B^3 = A$.

a. To make computation simpler, let's observe that $A = 9 \cdot \begin{bmatrix} 6 & 9 \\ -1 & 0 \end{bmatrix}$ and define $M = \begin{bmatrix} 6 & 9 \\ -1 & 0 \end{bmatrix}$. Now, $\chi_M(\lambda) = \lambda^2 - 6\lambda + 9$. Hence, $\lambda = 3$ is the only eigenvalue of M. If v is a corresponding eigenvector, then $M \cdot v = 3 \cdot v \Rightarrow \underbrace{9 \cdot M \cdot v}_{= A \cdot v} = 9 \cdot 3 \cdot v = 27 \cdot v$. Therefore, A's eigenvalue is 27 and we can use eigenvectors of M for the eigenvectors of A.

$\mathrm{Nul}(M - 3I) = \mathrm{Nul}\begin{bmatrix} 3 & 9 \\ -1 & -3 \end{bmatrix} = \mathrm{Span}\left\{\begin{bmatrix} 3 \\ -1 \end{bmatrix}\right\}$.

b. D is $\begin{bmatrix} 27 & 0 \\ 0 & 27 \end{bmatrix}$. $A - D = \begin{bmatrix} 27 & 81 \\ -9 & -27 \end{bmatrix} = N$. N is $9 \times \begin{bmatrix} 3 & 9 \\ -1 & -3 \end{bmatrix}$. Hence, $N^2 = 81\begin{bmatrix} 3 & 9 \\ -1 & -3 \end{bmatrix}\begin{bmatrix} 3 & 9 \\ -1 & -3 \end{bmatrix}$

$= 81\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0.$ $\blacksquare$

c. Consider something like this: $(\lambda I + P)^3$ for some matrix P s.t. $P^2 = 0$. Then, it is just $\underbrace{\lambda^3 I}_{\text{make this to be } D} + \underbrace{3\lambda^2 P}_{\text{make this to be } N} + 0 + 0$. So, $\lambda = 3$ and $P = N/3\lambda^2 = \frac{1}{27}N = \begin{bmatrix} 1 & 3 \\ -1/3 & -1 \end{bmatrix}$.

We now define $B = 3I + P = \begin{bmatrix} 4 & 3 \\ -1/3 & 2 \end{bmatrix}$.

Double check if this gives $B^3 = A$.

[1] Such a matrix is called *skew-symmetric*.
[2] An $n \times n$ matrix satisfying $N^n = 0$ is called *nilpotent*.

4. Consider the inner product space $\mathbb{P}_2$ with the inner product

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)dx.$$

a. Find an orthonormal basis for $\mathbb{P}_2$. [Hint. Use Gram-Schmidt process.]

b. Find the best approximation to $f(x) = x^5$ by polynomials in $\mathbb{P}_2$.

a. $\mathbb{P}_2$ has a basis $\{1, x, x^2\}$. Let's apply Gram-Schmidt process.

$v_1 = 1$, $v_2 = x - \frac{\langle 1, x\rangle}{\langle 1, 1\rangle} 1 = x - \frac{0}{2}\cdot 1 = x$, $v_3 = x^2 - \frac{\langle 1, x^2\rangle}{\langle 1, 1\rangle}\underbrace{1}_{v_1} - \frac{\langle x, x^2\rangle}{\langle x, x\rangle}\underbrace{x}_{v_2} = x^2 - \frac{2/3}{2}1 - \frac{0}{2/3}\cdot x = x^2 - 1/3$.

$\therefore \{1, x, x^2 - \frac{1}{3}\}$ is an orthogonal basis.

b. The best approximation is nothing but the projection. Applying the projection formula, we can just compute

$$\frac{\langle 1, x^5\rangle}{\langle 1, 1\rangle} 1 + \frac{\langle x, x^5\rangle}{\langle x, x\rangle} x + \frac{\langle x^2 - 1/3, x^5\rangle}{\langle x^2 - 1/3, x^2 - 1/3\rangle}(x^2 - 1/3) = \frac{0}{2}\cdot 1 + \frac{2/7}{2/3} x + \frac{\int_{-1}^{1}(x^2 - 1/3)x^5 dx}{\int_{-1}^{1}(x^2 - 1/3)^2 dx}(x^2 - 1/3)$$

However, $\int_{-1}^{1}(x^2 - 1/3)x^5 dx = \int_{-1}^{1}(x^7 - \frac{1}{3}x^5)dx = 0$ b/c $x^7 - \frac{1}{3}x^5$ is an odd function and the interval is from -1 to 1.

Therefore, the answer is $\frac{3}{7}x$.

5. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation such that

$$T\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}3\\5\end{bmatrix} \text{ and } T\begin{bmatrix}-1\\2\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix}.$$

a. Find a matrix A such that $Tv = Av$ for all $v \in \mathbb{R}^2$.

b. Given the basis $\mathcal{B} = \{\begin{bmatrix}1\\-1\end{bmatrix}, \begin{bmatrix}3\\3\end{bmatrix}\}$, find the matrix P such that $[T(v)]_{\mathcal{B}} = P[v]_{\mathcal{B}}$.

a. $A\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}3\\5\end{bmatrix}$ and $A\begin{bmatrix}-1\\2\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix}$. So, $A\begin{bmatrix}1 & -1\\1 & 2\end{bmatrix} = \begin{bmatrix}3 & 0\\5 & 1\end{bmatrix}$. Hence, $A = \begin{bmatrix}3 & 0\\5 & 1\end{bmatrix}\begin{bmatrix}1 & -1\\1 & 2\end{bmatrix}^{-1}$

$= \begin{bmatrix}3 & 0\\5 & 1\end{bmatrix}\cdot\frac{1}{3}\begin{bmatrix}2 & 1\\-1 & 1\end{bmatrix} = \begin{bmatrix}2 & 1\\3 & 2\end{bmatrix}$.

b. P's first column is obtained by $P e_1$. On the other hand $P e_1 = P[b_1]_{\mathcal{B}} = [T(b_1)]_{\mathcal{B}}$.

$T(b_1) = T(\begin{bmatrix}1\\-1\end{bmatrix}) = \begin{bmatrix}2 & 1\\3 & 2\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix} = \frac{1}{3}b_2$. So, $P e_1 = [\frac{1}{3}b_2]_{\mathcal{B}} = \frac{1}{3}e_2 = \begin{bmatrix}0\\1/3\end{bmatrix}$.

In a similar way, $P e_2 = [T(b_2)]_{\mathcal{B}} = \left[\begin{bmatrix}2 & 1\\3 & 2\end{bmatrix}\begin{bmatrix}3\\3\end{bmatrix}\right]_{\mathcal{B}} = \left[\begin{bmatrix}9\\15\end{bmatrix}\right]_{\mathcal{B}} = [-3b_1 + 4b_2]_{\mathcal{B}} = \begin{bmatrix}-3\\4\end{bmatrix}$.

$P = [Pe_1 \ Pe_2] = \begin{bmatrix}0 & -3\\1/3 & 4\end{bmatrix}$.

6. Let $M_{3\times 3}$ be the vector space of 3×3 real matrices. Let V be the set of matrices $X \in M_{3\times 3}$ such that $X^T = -X$. Is V a subspace? If so, find a basis for V.

- V is a subspace : 1) $0 \in V$. The zero matrix $0 = \begin{bmatrix} 0&0&0\\0&0&0\\0&0&0 \end{bmatrix}$ satisfies $0^T = -0$, so $0 \in V$.

  2) closed under addition. Suppose X and Y are in V. This means $X^T = -X$, $Y^T = -Y$.
  Then, $(X+Y)^T = X^T + Y^T = -X + (-Y) = -(X+Y)$.
  Hence, $X+Y \in V$. So, V is closed under addition.

  3) closed under scalar multiplication. Suppose $X \in V$ and choose any $c \in \mathbb{R}$.
  Then, $(c\cdot X)^T = c\cdot X^T = c\cdot(-X) = -(c\cdot X)$. $\Rightarrow c\cdot X \in V$.
  So, V is closed under scalar multiplication. $\blacksquare$

- A basis for V : Let's write X as $\begin{bmatrix} a&b&c\\d&e&f\\g&h&i \end{bmatrix}$. $X^T = -X$ implies $\begin{bmatrix} a&d&g\\b&e&h\\c&f&i \end{bmatrix} = \begin{bmatrix} -a&-b&-c\\-d&-e&-f\\-g&-h&-i \end{bmatrix}$.

  So, $a=-a$, $e=-e$, $i=-i$ $\Rightarrow a=e=i=0$.
  For d, g, h, b, c, f, they should be negative pairwise. So, only free variable is b, c, f and $d=-b$, $g=-c$, $h=-f$ follows.
  $\Rightarrow \left\{ \begin{bmatrix} 0&1&0\\-1&0&0\\0&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&1\\0&0&0\\-1&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&0\\0&0&1\\0&-1&0 \end{bmatrix} \right\}$ is a basis.

7. Let
$$F = \begin{bmatrix} -2 & -1 \\ 4 & 3 \end{bmatrix}.$$

Let $M_{2\times 2}$ be the vector space of 2×2 real matrices and consider the map $T : M_{2\times 2} \to M_{2\times 2}$ defines as $T(X) = XF$ for any $X \in M_{2\times 2}$. Find the matrix of the linear transformation T with respect to the basis $\mathcal{B} = \{b_1, b_2, b_3, b_4\}$ where

$$b_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, b_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, b_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, b_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

The matrix of the linear transformation T w.r.t the basis is computed as
$$\left[[T(b_1)]_{\mathcal{B}} \; [T(b_2)]_{\mathcal{B}} \; [T(b_3)]_{\mathcal{B}} \; [T(b_4)]_{\mathcal{B}} \right]$$

$T(b_1) = b_1 \cdot F = \begin{bmatrix} 1&0\\0&0 \end{bmatrix} \cdot \begin{bmatrix} -2&-1\\4&3 \end{bmatrix} = \begin{bmatrix} -2&-1\\0&0 \end{bmatrix} = -2b_1 + (-1)b_2 + 0\cdot b_3 + 0\cdot b_4 \Rightarrow [T(b_1)]_{\mathcal{B}} = \begin{bmatrix} -2\\-1\\0\\0 \end{bmatrix}$.

$T(b_2) = b_2 \cdot F = \begin{bmatrix} 4&3\\0&0 \end{bmatrix} = 4b_1 + 3b_2 + 0\cdot b_3 + 0\cdot b_4 \Rightarrow [T(b_2)]_{\mathcal{B}} = \begin{bmatrix} 4\\3\\0\\0 \end{bmatrix}$.

Similarly, $[T(b_3)]_{\mathcal{B}} = \begin{bmatrix} 0\\0\\-2\\-1 \end{bmatrix}$ and $[T(b_4)]_{\mathcal{B}} = \begin{bmatrix} 0\\0\\4\\3 \end{bmatrix}$.

Therefore, the matrix is
$$\begin{bmatrix} -2&4&0&0\\-1&3&0&0\\0&0&-2&4\\0&0&-1&3 \end{bmatrix}.$$

8. Show that the functions $\{\sin(x/2), \sin(3x/2), \cdots, \sin((2n+1)x/2), \cdots\}$ are orthogonal under the inner product

$$\langle f(x), g(x)\rangle = \int_0^\pi f(x)g(x)dx.$$

For any $n > 1$, find the orthogonal projection $\mathcal{J}_n(x)$ of the function $f(x) = x$ onto the subspace spanned by $\{\sin(x/2), \sin(3x/2), \cdots, \sin((2n+1)x/2)\}$.

- Orthogonality of $\{\sin \frac{2n+1}{2}x\}$ $(n=0,1,\cdots)$

What we need to check: $\langle \sin\frac{2m+1}{2}x, \sin\frac{2n+1}{2}x\rangle = 0$ if $m \neq n$.

Computation: It is $\int_0^\pi \sin\frac{2m+1}{2}x \sin\frac{2n+1}{2}x$. We use $\sin\alpha\sin\beta = [\cos(\alpha+\beta) - \cos(\alpha-\beta)]/2$.

$$= \int_0^\pi \frac{1}{2}\left[\cos\left(\frac{2m+1}{2}+\frac{2n+1}{2}\right)x - \cos\left(\frac{2m+1}{2}-\frac{2n+1}{2}\right)x\right]dx$$

$$= \frac{1}{2}\int_0^\pi \cos(m+n+1)x\,dx - \frac{1}{2}\int_0^\pi \cos(m-n)x\,dx$$

$$= \frac{1}{2}\frac{1}{m+n+1}\sin(m+n+1)x\Big|_0^\pi - \frac{1}{2}\underbrace{\frac{1}{m-n}}_{\text{nonzero since } m\neq n}\sin(m-n)x\Big|_0^\pi = \frac{1}{2}(0-0) - \frac{1}{2}(0-0) = 0$$

because $\sin k\pi = 0$ for k: integer. $\blacksquare$

- The orthogonal projection of $f(x)=x$ onto $\mathrm{Span}\{\sin\frac{1}{2}x, \cdots, \sin\frac{2n+1}{2}x\}$.

We checked that $\sin\frac{2n+1}{2}x$'s form an orthogonal set, so we can use the projection formula.

$$\mathcal{J}_n(x) = \frac{\langle \sin\frac{x}{2}, x\rangle}{\langle \sin\frac{x}{2}, \sin\frac{x}{2}\rangle}\sin\frac{x}{2} + \cdots + \frac{\langle \sin\frac{2n+1}{2}x, x\rangle}{\langle \sin\frac{2n+1}{2}x, \sin\frac{2n+1}{2}x\rangle}\sin\frac{2n+1}{2}x.$$

Let's compute $\langle \sin\frac{2n+1}{2}x, \sin\frac{2n+1}{2}x\rangle$. It is $\int_0^\pi \sin^2\frac{2n+1}{2}x\,dx = \int_0^\pi \frac{1}{2}[1-\cos(2n+1)x]\,dx$

$$= \frac{1}{2}x\Big|_0^\pi - \frac{1}{2}\cdot\frac{\sin(2n+1)x}{2n+1}\Big|_0^\pi$$

$$= \pi/2 - \frac{1}{2}(0-0) = \pi/2.$$

Now, let's compute $\langle \sin\frac{2n+1}{2}x, x\rangle$. It is $\int_0^\pi x\sin\frac{2n+1}{2}x\,dx$.

Substitute $\frac{2n+1}{2}x$ by t. $x = \frac{2}{2n+1}t$ and $dx = \frac{2}{2n+1}dt$.

So, the integration is just $\left(\frac{2}{2n+1}\right)^2\int_0^{\frac{2n+1}{2}\pi} t\sin t\,dt$. Now, apply integration by parts.

$\left(\int t\sin t\,dt = t\cdot -\cos t - \int 1\cdot(-\cos t)\,dt = -t\cos t + \sin t.\right)$

$$\Rightarrow \left(\frac{2}{2n+1}\right)^2\int_0^{\frac{2n+1}{2}\pi} t\sin t\,dt = \left(\frac{2}{2n+1}\right)^2\left[-t\cos t + \sin t\right]_0^{\frac{2n+1}{2}\pi} = \left(\frac{2}{2n+1}\right)^2\left((0+(-1)^n) - (0+0)\right) = \frac{4\cdot(-1)^n}{(2n+1)^2}$$

Therefore, $\mathcal{J}_n(x) = \frac{8}{\pi}\left(\sin\frac{x}{2} - \frac{1}{9}\sin\frac{3}{2}x + \frac{1}{25}\sin\frac{5}{2}x - \cdots + (-1)^n\frac{1}{(2n+1)^2}\sin\frac{2n+1}{2}x\right)$.