1. Suppose that B is an invertible square matrix with the property that for both B and B^{-1}, all of their entries are integers. Show that $\det B$ is 1 or -1.

2. Let B be an $n \times n$ matrix satisfying $B^T = -B$.[1] By considering the determinant, show that B is not invertible if n is odd.

3. Let
$$A = \begin{bmatrix} 54 & 81 \\ -9 & 0 \end{bmatrix}.$$
   a. Find the eigenvalues of A and the corresponding eigenvector.
   b. Let D be the diagonal matrix whose diagonal entries are exactly the eigenvalues from the above. Check that N which is defined to be $A - D$ satisfies $N^2 = 0$.[2]
   c. Use $N^2 = 0$ to find a matrix B such that $B^3 = A$.

4. Consider the inner product space $\mathbb{P}_2$ with the inner product
$$\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)dx.$$
   a. Find an orthonormal basis for $\mathbb{P}_2$. [Hint. Use Gram-Schmidt process.]
   b. Find the best approximation to $f(x) = x^5$ by polynomials in $\mathbb{P}_2$.

5. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation such that
$$T\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix} \text{ and } T\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$
   a. Find a matrix A such that $Tv = Av$ for all $v \in \mathbb{R}^2$.
   b. Given the basis $\mathcal{B} = \{\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \end{bmatrix}\}$, find the matrix P such that $[T(v)]_{\mathcal{B}} = P[v]_{\mathcal{B}}$.

6. Let $M_{3\times 3}$ be the vector space of 3×3 real matrices. Let V be the set of matrices $X \in M_{3\times 3}$ such that $X^T = -X$. Is V a subspace? If so, find a basis for V.

7. Let
$$F = \begin{bmatrix} -2 & -1 \\ 4 & 3 \end{bmatrix}.$$
   Let $M_{2\times 2}$ be the vector space of 2×2 real matrices and consider the map $T : M_{2\times 2} \to M_{2\times 2}$ defines as $T(X) = XF$ for any $X \in M_{2\times 2}$. Find the matrix of the linear transformation T with respect to the basis $\mathcal{B} = \{b_1, b_2, b_3, b_4\}$ where
$$b_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, b_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, b_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, b_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

8. Show that the functions $\{\sin(x/2), \sin(3x/2), \cdots, \sin((2n+1)x/2), \cdots\}$ are orthogonal under the inner product
$$\langle f(x), g(x) \rangle = \int_{0}^{\pi} f(x)g(x)dx.$$
   For any $n > 1$, find the orthogonal projection $\mathcal{J}_n(x)$ of the function $f(x) = x$ onto the subspace spanned by $\{\sin(x/2), \sin(3x/2), \cdots, \sin((2n+1)x/2)\}$.

[1] Such a matrix is called *skew-symmetric*.
[2] An $n \times n$ matrix satisfying $N^n = 0$ is called *nilpotent*.