1. Mark "T" if the statement is always true, "F" if it is sometimes false. ***No*** *explanations are needed.*

T | If W and W' are subspaces of a vector space V, the set of vectors in V that belong to both W and W' is a subspace of V.

F | If A is similar to B and B is orthogonal then A must be orthogonal.

T | For every subspace H of $\mathbb{R}^n$, there is a matrix A such that $H = \text{Nul}A$.

F | If λ is an eigenvalue of A and μ is an eigenvalue of B and both are $n \times n$, then $\lambda\mu$ must be an eigenvalue of AB.

F | The normal equation $A^T A\mathbf{x} = A^T\mathbf{b}$ always have a unique solution.

T | The change-of-coordinates matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ between the bases $\mathcal{B} = \{2e_1, 3e_2\}$ and $\mathcal{C} = \{-3e_1, 4e_2\}$ of $\mathbb{R}^2$ is a diagonal matrix.

T | There exists a real 3×3 matrix of rank 2 with only one distinct eigenvalue.

F | The set of vectors $\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T$ so that $x_1 + x_2 + x_3 \geq 0$ forms a subspace of $\mathbb{R}^3$.

F | If V is a vector space, and H_1 and H_2 are subspaces, then the union of H_1 and H_2 (i.e. the set of vectors that lie in H_1 or H_2) is always a subspace.

F | The dimensions of the column space and of the nullspace of a matrix add up to the number of rows.

T | Suppose that A is a symmetric $n \times n$ matrix and W is a subspace of $\mathbb{R}^n$ such that $Aw \in W$ for all $w \in W$. Then, $Av \in W^\perp$ for all $v \in W^\perp$ where $W^\perp$ is the orthogonal complement of W with respect to the dot product.

F | Suppose that $v_1(t)$, $v_2(t)$ are vector functions taking values in $\mathbb{R}^2$. If the Wronskian $W[v_1, v_2](t)$ is equal to 0 for all $t \in \mathbb{R}$, then $v_1(t)$, $v_2(t)$ are linearly dependent.

F | The set of solutions to $ay'' + by' + cy = 0$ is a two-dimensional vector space for any a, b, $c \in \mathbb{R}$.

F | The eigenvalues of an orthogonal matrix are all real.

T | For any matrix A, the matrix AA^T is diagonalizable.

T | The function $y(t) = t\sin t$ is a solution to $y'''' + 2y'' + y = 0$.

2. Select the correct answers. Be aware that there might be more than one answer to each problem.

 1) A number λ is an eigenvalue of an $n \times n$ matrix A if and only if:
 (a) $\det(A - \lambda I_n) = 0$.
 (b) λ is a pivot of A.
 (c) $A - \lambda I_n$ is invertible.
 (d) $A\mathbf{x} = \lambda \mathbf{x}$ for some $\mathbf{x} \in \mathbb{R}^n$.

 2) A collection of n vectors $\mathbf{v}_1, \cdots, \mathbf{v}_n$ in $\mathbb{R}^n$ is linearly independent if and only if:
 (a) It forms a basis of $\mathbb{R}^n$.
 (b) Any two vectors in it are linearly independent.
 (c) $\mathbf{v}_1 + \cdots + \mathbf{v}_n = \mathbf{0}$ implies that each $\mathbf{v}_i = \mathbf{0}$.
 (d) $c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n = \mathbf{0}$ implies that each $c_i = 0$.

 3) The rank of a matrix A is
 (a) The number of rows.
 (b) The dimension of its column space.
 (c) The dimension of its row space.
 (d) $\dim(\text{Row } A)^\perp$.

3. Find the general solution to the equation

$$y'' + 3y' + 2y = \frac{1}{e^t + 1}.$$

Homogeneous case solution: $y_1(t) = e^{-2t}$, $y_2(t) = e^{-t}$.

Use variation of parameters: $v_1(t) = \int \frac{-y_2 g}{W}$ and $v_2 = \int \frac{y_1 g}{W}$. $W = y_1y_2' - y_1'y_2 = e^{-2t}\cdot(-1)e^{-t} - (-2)e^{-2t}\cdot e^{-t} = e^{-3t}$.

So, $v_1(t) = \int -\frac{e^{-t}\cdot\frac{1}{e^t+1}}{e^{-3t}}dt = \int \frac{-e^{2t}}{e^t+1}dt = \int -\frac{s}{s+1}ds$ ($s = e^t$, $ds = e^t dt = s\,dt$)

$= \int\left(-1 + \frac{1}{s+1}\right)ds = -s + \ln(s+1) = -e^t + \ln(e^t+1)$.

$v_2(t) = \int \frac{e^{-2t}\cdot\frac{1}{e^t+1}}{e^{-3t}}dt = \int \frac{e^t}{e^t+1}dt = \ln(e^t+1)$ (b/c $\int \frac{f'}{f} = \ln f$)

General solution: $(-e^t + \ln(e^t+1))\cdot e^{-2t} + \ln(e^t+1)\cdot e^{-t} + c_1e^{-2t} + c_2e^{-t}$.

4. Find the general solution to the following system:

$$\mathbf{x}'(t) = \begin{bmatrix} 1 & -6 \\ 1 & -4 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 2 \\ 1 \end{bmatrix} te^{-3t}.$$

Homogeneous case solution : Find eigenvalues and eigenvectors.

$A = \begin{bmatrix} 1 & -6 \\ 1 & -4 \end{bmatrix}$. $\chi_A(\lambda) = \det(A - \lambda I) = (\lambda-1)(\lambda+4)+6 = \lambda^2+3\lambda+2.$

$\lambda = -1 \rightarrow \text{Nul}\begin{bmatrix} 2 & -6 \\ 1 & -3 \end{bmatrix} \ni \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ $\Longrightarrow$ $\mathbf{x}_1(t) = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \cdot e^{-t}$

$-2 \rightarrow \text{Nul}\begin{bmatrix} 3 & -6 \\ 1 & -2 \end{bmatrix} \ni \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. $\mathbf{x}_2(t) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \cdot e^{-2t}$

Particular solution : Method of Undetermined Coefficients

$\mathbf{x}_p(t) = (ut+v) \cdot e^{-3t}$ b/c $\begin{bmatrix} 2 \\ 1 \end{bmatrix} t \cdot e^{-3t}$ is (poly of deg 1) × exponential.

$(\text{LHS}) = u \cdot e^{-3t} + (-3)(ut+v)e^{-3t}$ $(\text{RHS}) = A(ut+v)e^{-3t} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} t e^{-3t}$.

Cancel out e^{-3t} and look at the terms w/ t and w/o t separately.

w/ t: $-3u = Au + \begin{bmatrix} 2 \\ 1 \end{bmatrix} \Rightarrow (A+3I)u = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$ $u = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$.

w/o t: $u - 3v = Av \Longrightarrow v = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$.

General solution : $\left(\begin{bmatrix} -2 \\ -1 \end{bmatrix} t + \begin{bmatrix} -2 \\ -1 \end{bmatrix}\right) e^{-3t} + C_1 \begin{bmatrix} 3 \\ 1 \end{bmatrix} e^{-t} + C_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-2t}$.

5. The system $\mathbf{x}'(t) = \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix} \mathbf{x}(t)$ has a solution $\mathbf{x}_1(t) = e^{2t} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Find another solution which is linearly independent to $\mathbf{x}_1(t)$. (Hint. Try $\mathbf{x}_2(t) = te^{2t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + e^{2t} v$.)

Try $\mathbf{x}_2(t) = t \cdot e^{2t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + e^{2t} \cdot v$.

(LHS) $e^{2t}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2te^{2t}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2e^{2t}v$ (RHS) $A\begin{bmatrix} 1 \\ 0 \end{bmatrix} te^{2t} + Av \cdot e^{2t}$ where $A = \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}$.

Similarly to #4, we look at the terms w/ t and w/o t separately.

w/ t : $2 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = A \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

w/o t : $\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2v = Av$ $\Longrightarrow v = \begin{bmatrix} 0 \\ 1/3 \end{bmatrix}$ is one solution.

So, $\mathbf{x}_2(t) = te^{2t}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + e^{2t}\begin{bmatrix} 0 \\ 1/3 \end{bmatrix}$.

6. Find the values of the positive number λ for which the given problem below has a nontrivial solution.

$$y'' + \lambda y = 0 \text{ for } 0 < x < \pi; \quad y(0) = 0, \quad y'(\pi) = 0.$$

For the sake of simplicity, let β be $\sqrt{\lambda}$.

The auxiliary equation is $r^2 + \lambda = 0$. $\Rightarrow r = \pm\sqrt{\lambda}\, i = \pm\beta i$. Hence, $y_1(t) = \cos\beta t$, $y_2(t) = \sin\beta t$. hom-case solutions.

Let $y(t) = C_1 y_1(t) + C_2 y_2(t)$.

$0 = y(0) = C_1 \cdot 1 + C_2 \cdot 0 = C_1$. So, $C_1 = 0$. $\Rightarrow y(t) = C_2 y_2(t)$.

Now, $\underset{\underset{0}{\parallel}}{y'(\pi)} = C_2 \cdot \beta \cos\beta\pi$. But we want a non-trivial solution. So, $C_2 \neq 0$ and β is $\sqrt{\lambda} > 0$. So, $\cos\beta\pi = 0$ which implies that $\beta = \frac{2n+1}{2}$ (n: integer).

$\Rightarrow \lambda = \frac{(2n+1)^2}{4}$ (n: integer).

Namely, $\lambda = \frac{1}{4}, \frac{9}{4}, \frac{25}{4}, \frac{49}{4}, \ldots$ are only possible λ's.

7. Suppose that $f(x) = 0$ for $-\pi < x < 0$ and $f(x) = 1$ for $0 \le x \le \pi$. Find the Fourier series for $f(x)$ on $[-\pi, \pi]$.[1]

Fourier series is $\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$ where $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx$

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx$.

So, $a_n = \frac{1}{\pi}\int_0^{\pi} \cos nx\, dx = \frac{1}{\pi} \left.\frac{\sin nx}{n}\right|_0^{\pi}$ if $n \neq 0$. ($n=0$ case $\Rightarrow \frac{1}{\pi}\int_0^{\pi} 1\, dx = 1$.)

$= \frac{1}{\pi} \cdot 0 = 0$.

$b_n = \frac{1}{\pi}\int_0^{\pi} \sin nx\, dx = \frac{1}{\pi} \cdot -\left.\frac{\cos nx}{n}\right|_0^{\pi} = \frac{1}{n\pi}\left(1 - (-1)^n\right) = \begin{cases} \frac{2}{n\pi} & n: \text{odd} \\ 0 & n: \text{even}. \end{cases}$

So, the Fourier series is $\frac{2}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right)$.

[1] Optional.