1. Mark "T" if the statement is always true, "F" if it is sometimes false. ***No** explanations are needed.*

T F | If $W$ and $W'$ are subspaces of a vector space $V$, the set of vectors in $V$ that belong to both $W$ and $W'$ is a subspace of $V$.

T F | If $A$ is similar to $B$ and $B$ is orthogonal then $A$ must be orthogonal.

T F | For every subspace $H$ of $\mathbb{R}^n$, there is a matrix $A$ such that $H = \text{Nul}A$.

T F | If $\lambda$ is an eigenvalue of $A$ and $\mu$ is an eigenvalue of $B$ and both are $n \times n$, then $\lambda\mu$ must be an eigenvalue of $AB$.

T F | The normal equation $A^T A\mathbf{x} = A^T\mathbf{b}$ always have a unique solution.

T F | The change-of-coordinates matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ between the bases $\mathcal{B} = \{2e_1, 3e_2\}$ and $\mathcal{C} = \{-3e_1, 4e_2\}$ of $\mathbb{R}^2$ is a diagonal matrix.

T F | There exists a real $3 \times 3$ matrix of rank 2 with only one distinct eigenvalue.

T F | The set of vectors $\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T$ so that $x_1 + x_2 + x_3 \geq 0$ forms a subspace of $\mathbb{R}^3$.

T F | If $V$ is a vector space, and $H_1$ and $H_2$ are subspaces, then the union of $H_1$ and $H_2$ (i.e. the set of vectors that lie in $H_1$ or $H_2$) is always a subspace.

T F | The dimensions of the column space and of the nullspace of a matrix add up to the number of rows.

T F | Suppose that $A$ is a symmetric $n \times n$ matrix and $W$ is a subspace of $\mathbb{R}^n$ such that $Aw \in W$ for all $w \in W$. Then, $Av \in W^\perp$ for all $v \in W^\perp$ where $W^\perp$ is the orthogonal complement of $W$ with respect to the dot product.

T F | Suppose that $v_1(t), \cdots, v_n(t)$ are vector functions taking values in $\mathbb{R}^n$. If the Wronskian $W[v_1, \cdots, v_n](t)$ is equal to 0 for all $t \in \mathbb{R}$, then $v_1(t), \cdots, v_n(t)$ are linearly dependent.

T F | The set of solutions to $ay'' + by' + cy = 0$ is a two-dimensional vector space for any $a$, $b$, $c \in \mathbb{R}$.

T F | The eigenvalues of an orthogonal matrix are all real.

T F | For any matrix $A$, the matrix $AA^T$ is diagonalizable.

T F | The function $y(t) = t\sin t$ is a solution to $y'''' + 2y'' + y = 0$.

2. Select the correct answers. Be aware that there might be more than one answer to each problem.

1) A number $\lambda$ is an eigenvalue of an $n \times n$ matrix $A$ if and only if:
(a) $\det(A - \lambda I_n) = 0$.
(b) $\lambda$ is a pivot of $A$.
(c) $A - \lambda I_n$ is invertible.
(d) $A\mathbf{x} = \lambda\mathbf{x}$ for some $\mathbf{x} \in \mathbb{R}^n$.

2) A collection of $n$ vectors $\mathbf{v}_1, \cdots, \mathbf{v}_n$ in $\mathbb{R}^n$ is linearly independent if and only if:
(a) It forms a basis of $\mathbb{R}^n$.
(b) Any two vectors in it are linearly independent.
(c) $\mathbf{v}_1 + \cdots + \mathbf{v}_n = \mathbf{0}$ implies that each $\mathbf{v}_i = \mathbf{0}$.
(d) $c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n = \mathbf{0}$ implies that each $c_i = 0$.

3) The rank of a matrix $A$ is
(a) The number of rows.
(b) The dimension of its column space.
(c) The dimension of its row space.
(d) $\dim(\text{Row } A)^\perp$.

3. Find the general solution to the equation

$$y'' + 3y' + 2y = \frac{1}{e^t + 1}.$$

4. Find the general solution to the following system:

$$\mathbf{x}'(t) = \begin{bmatrix} 1 & -6 \\ 1 & -4 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 2 \\ 1 \end{bmatrix} te^{-3t}.$$

5. The system $\mathbf{x}'(t) = \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix} \mathbf{x}(t)$ has a solution $\mathbf{x}_1(t) = e^{2t} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Find another solution which is linearly independent to $\mathbf{x}_1(t)$. (Hint. Try $\mathbf{x}_2(t) = te^{2t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + e^{2t} v$.)

6. Find the values of the positive number $\lambda$ for which the given problem below has a nontrivial solution.

$$y'' + \lambda y = 0 \text{ for } 0 < x < \pi; \quad y(0) = 0, \quad y'(\pi) = 0.$$

7. Suppose that $f(x) = 0$ for $-\pi < x < 0$ and $f(x) = 1$ for $0 \leq x \leq \pi$. Find the Fourier series for $f(x)$ on $[-\pi, \pi]$.[1]

[1] Optional.