Optional *(plaese check out the videos on my webpage)*

1 Fourier Sine and Cosine Series

The reason we are studying Fourier Series in this course is to introduce a way to solve Heat Equations using Fourier series. However, in fact, we cannot use the original Fourier series but need a slightly different version of that.

1.1 Some Extensions

Let $f(x)$ be a function defined on $(0, L)$. We have two extensions of this function: **odd $2L$-periodic extension** and **even $2L$-periodic extension**. Geometrically, they are just rotation by π around 0 and reflection with respect to y-axis.

1.2 Fourier Cosine and Sine Series

Let $f(x)$ be piecewise continuous on the interval $[0, L]$. The Fourier cosine series of $f(x)$ on $[0, L]$ is

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}, \qquad \text{where} \qquad a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx.$$

The Fourier sine series of $f(x)$ on $[0, L]$ is

$$\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}, \qquad \text{where} \qquad b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx.$$

2 Heat Equation

2.1 A Model for Heat Flow

One-dimensional heat flow equation with **boundary conditions** and **initial condition** can be written as follows:

Boundary Temperature 0

$$\begin{aligned}\frac{\partial u}{\partial t}(x,t) &= \beta\frac{\partial^2 u}{\partial x^2}(x,t), \quad 0 < x < L, \quad t > 0,\\ u(0,t) &= u(L,t) = 0, \quad t > 0,\\ u(x,0) &= f(x), \quad 0 < x < L.\end{aligned}$$

The second conditions are the conditions that 'at any time t, the temperature of the ends of the wire is 0. The third condition is 'the distribution of heat at time $t = 0$'.

Boundary Heat Flow 0

$$\begin{aligned}\frac{\partial u}{\partial t}(x,t) &= \beta\frac{\partial^2 u}{\partial x^2}(x,t), \quad 0 < x < L, \quad t > 0,\\ \frac{\partial u}{\partial x}(0,t) &= \frac{\partial u}{\partial x}(L,t) = 0, \quad t > 0,\\ u(x,0) &= f(x), \quad 0 < x < L.\end{aligned}$$

This equation is a modified version. Now, the boundary condition is that there is no heat flow going out or coming in.

2.2 Separation of Variables

Try

$$u(x,t) = X(x)T(t).$$

Then, one gets $u_n(x,t) = e^{-\beta(n\pi/L)^2 t}\sin\frac{n\pi x}{L}$ are satisfying the first and second condition. Now, set up

$$u(x,t) = \sum_{n=1}^{\infty} c_n u_n(x,t)$$

and solve for c_n's.

Then, one gets $u_n(x,t) = e^{-\beta(n\pi/L)^2 t}\cos\frac{n\pi x}{L}$ are satisfying the first and second condition. Now, set up

$$u(x,t) = \sum_{n=0}^{\infty} c_n u_n(x,t)$$

and solve for c_n's.